International Journal of Pure and Applied Mathematics
Volume 60 No. 3 2010, 355-362

INCREMENT OF THE OBJECTIVE FUNCTION AND OPTIMALITY CRITERION FOR ONE NON-HOMOGENEOUS NETWORK FLOW PROGRAMMING PROBLEM

L.A. Pilipchuk[1], A.S. Pilipchuk[2], Y.H. Pesheva[3 §]
[1]Belarussian State University
4, Nezalezhnosti Ave., Minsk, 220050, BELARUS
e-mail: pilipchuk@bsu.by
[2]State Scientific Institution
99, Academician A.K. Krasin Str., Minsk, 220109, BELARUS
e-mail: pilip74@rambler.ru
[3]Department of Differential Equations
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
P.O. Box 384, Sofia, 1000, BULGARIA
e-mail: yhp@vmei.acad.bg

Abstract: For an linear non-homogeneous flow programming problem with additional constraints of general kind are obtained the increment of the objective function using network properties of the problem and principles of decomposition of a support. Optimality conditions are received.

AMS Subject Classification: 65K05, 90C08, 90C35
Key Words: network flow, non-homogeneous problem, support, objective function, criterion of optimality, nonsingular plan

Received: March 25, 2010

[§]Correspondence author

1. Statement of the Problem, Basic Concepts and Definitions

Consider the following mathematical model of inhomogeneous extreme network flow problem

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} c_{ij}^k x_{ij}^k \longrightarrow \min, \tag{1}$$

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K; \tag{2}$$

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = \overline{1,l}; \tag{3}$$

$$\sum_{k\in K_0(i,j)} x_{ij}^k \le d_{ij}^0, \quad \text{for } (i,j) \in U_0; \tag{4}$$

$$0 \le x_{ij}^k \le d_{ij}^k, \quad \text{for } k \in K_1(i,j), (i,j) \in U; \tag{5}$$

$$x_{ij}^k \ge 0, \quad \text{for } k \in K(i,j) \setminus K_1(i,j), (i,j) \in U, \tag{6}$$

where $G = (I, U)$ – a finite orientated connected network without multiple arcs and loops, I is a set of nodes and $U \subset I \times I$ is a set of arcs; $K = \{1, \ldots, |K|\}$ – a finite non-empty set of different products (commodities) is transported through the network G. For each $k \in K$ there exists a connected subnetwork $G^k = (I^k, U^k) \subseteq G$, such that $U^k \subseteq U$ is a non-empty set of arcs carrying the k-th product, $I^k = I(U^k)$ – is the set of nodes used for transporting (producing/consuming/transiting) the k-th product. In order to distinguish the products, which can simultaneously pass through an arc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j) \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$. Now let us define a set $U_0 \subseteq U$ as an arbitrary subset of multiarcs of the network G. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j), |K_0(i,j)| > 1$. For all arcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be non-negative. Moreover, each arc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the arc (i,j). $I_i^+(U^k) = \{j \in I^k : (i,j) \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j,i) \in U^k\}$; x_{ij}^k – amount of the k-th product transported through an arc (i,j); c_{ij}^k – transportation cost through an arc (i,j) of a unit of the k-th product; d_{ij}^k – carrying capacity of an arc (i,j) for the k-th product; d_{ij}^0 – aggregate capacity of an arc $(i,j) \in U_0$ for a total amount of products $K_0(i,j)$; λ_{ij}^{kp} – weight of a unit of the k-th product transported through an arc (i,j) in the p-th additional constraint; α_p – total weighted amount of products imposed by the p-th additional constraint; a_i^k – intensity of a node i for the k-th product.

2. Formula for Increment of the Objective Function

Let $x = (x_{ij}^k, (i,j) \in U, k \in K(i,j))$ be a plan [2] of the problem (1)-(6), i.e. components of the vector x meet the conditions (2)-(6). Along with the plan x let us define support plan $\{x, U_S\}$ as a pair, containing of an arbitrary flow x and a support $U_S = \{U_S^k, k \in K; U^*\}$ $U^* \subseteq \overline{U}_0, \overline{U}_0 = \{(i,j) \in U_0 : |K_S^0(i,j)| > 1\}$ of the problem (1)-(6) [2, 4]. Let us consider some other plan $\overline{x} = (\overline{x}_{ij}^k : (i,j) \in U,\ k \in K(i,j)) = (x_{ij}^k + \Delta x_{ij}^k : (i,j) \in U, k \in K(i,j))$. Then $\Delta x = \Big(\Delta x_{ij}^k, (i,j) \in U, k \in K(i,j)\Big)$ is the vector of flow increments along the arc $(i,j) \in U$.

Let us denote

$$z_{ij} = \sum_{k \in K_0(i,j)} x_{ij}^k, \quad \overline{z}_{ij} = \sum_{k \in K_0(i,j)} \overline{x}_{ij}^k,$$
$$\Delta z_{ij} = \overline{z}_{ij} - z_{ij} = \sum_{k \in K_0(i,j)} \Delta x_{ij}^k,\ (i,j) \in U_0. \tag{7}$$

Since the plan $\overline{x}$ meets the conditions (2)-(6) then the following relations are true

$$\sum_{j \in I_i^+(U^k)} \overline{x}_{ij}^k - \sum_{j \in I_i^-(U^k)} \overline{x}_{ji}^k = a_i^k,\ i \in I^k,\ k \in K, \tag{8}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} \overline{x}_{ij}^k = \alpha^p,\ p = \overline{1, l}, \tag{9}$$

$$\sum_{k \in K_0(i,j)} \overline{x}_{ij}^k \le d_{ij}^0,\ \overline{x}_{ij}^k \ge 0,\ k \in K_0(i,j),\ (i,j) \in U^*, \tag{10}$$

where the constraints (4) are written down only for the support multiarcs U^*.

Subtracting from (8)-(10) the corresponding constraints (2)-(4), we obtain:

$$\sum_{j \in I_i^+(U^k)} \Delta x_{ij}^k - \sum_{j \in I_i^-(U^k)} \Delta x_{ij}^k = 0,\ i \in I^k,\ k \in K, \tag{11}$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^k \Delta x_{ij}^k = 0,\ p = \overline{1, l}, \tag{12}$$

$$\sum_{k\in K_0(i,j)} \Delta x_{ij}^k = \Delta z_{ij},\ (i,j)\in U^*, \tag{13}$$

where Δz_{ij} is defined by formula (7).

Let us order components of solution of system (11)-(13) the following way: $\Delta x' = (\Delta x'_T, \Delta x'_C, \Delta x'_N)$, where $\Delta x'_T = (\Delta x^k_{ij}, (i,j)^k \in U^k_T, k\in K)$, $\Delta x'_C = (\Delta x^k_{ij}, (i,j)^k \in U^k_C, k\in K)$, $\Delta x'_N = (\Delta x^k_{ij}, (i,j)^k \in U^k_N, k\in K)$, $U^k_N = U^k \backslash \left(U^k_T \bigcup U^k_C\right)$, U^k_T – spanning tree of the graph G^k, $k\in K$.

The general solution of the homogeneous system (11) is the following [4]:

$$\Delta x^k_{ij} = \sum_{(\tau,\rho)^k\in U^k\backslash U^k_T} \Delta x^k_{\tau\rho} \mathrm{sign}(i,j)^{L^k_{t(\tau,\rho)}}, (i,j)^k\in U^k_T, k\in K, \tag{14}$$

$$\mathrm{sign}(i,j)^{L^k_t} = \begin{cases} 1, & \text{if} \quad (i,j)^k\in L^{k+}_t; \\ -1, & \text{if} \quad (i,j)^k\in L^{k-}_t; \\ 0, & \text{if} \quad (i,j)^k\notin L^k_t. \end{cases}$$

Let us put the items, corresponding to components of the vector $\Delta x'_T$, together:

$$\Delta\varphi(x) = \sum_{k\in K}\sum_{(i,j)^k\in U^k} c^k_{ij}\Delta x^k_{ij} = \sum_{k\in K}\sum_{(i,j)^k\in U^k_T} c^k_{ij}\Delta x^k_{ij} + \sum_{k\in K}\sum_{(i,j)^k\in U^k\backslash U^k_T} c^k_{ij}\Delta x^k_{ij}. \tag{15}$$

Let us substitute (14) into (15):

$$\Delta\varphi(x) = \sum_{k\in K}\sum_{(i,j)^k\in U^k_T} c^k_{ij}\left[\sum_{(\tau,\rho)^k\in U^k\backslash U^k_T}\Delta x^k_{\tau\rho}\mathrm{sign}(i,j)^{L^k_{t(\tau,\rho)}}\right] + \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U^k_T} c^k_{\tau\rho}\Delta x^k_{\tau\rho}$$

$$= \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U^k_T}\left[c^k_{\tau\rho} + \sum_{(i,j)^k\in U^k_T} c^k_{ij}\mathrm{sign}(i,j)^{L^k_{t(\tau,\rho)}}\right]\Delta x^k_{\tau\rho}.$$

Let us denote $\sum\limits_{(i,j)^k\in L^k_{t(\tau,\rho)}} c^k_{ij}\mathrm{sign}(i,j)^{L^k_{t(\tau,\rho)}}$, with $\widetilde{\Delta}^k_{\tau\rho}$. Then

$$\Delta\varphi(x) = \sum_{k\in K}\sum_{(\tau,\rho)^k\in U^k\backslash U^k_T}\widetilde{\Delta}^k_{\tau\rho}\Delta x^k_{\tau\rho}. \tag{16}$$

Knowing that $U^k \backslash U_T^k = U_C^k \bigcup U_N^k$, we break the sum again:

$$\Delta\varphi(x) = \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_C^k} \widetilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k + \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_N^k} \widetilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k. \quad (17)$$

By analogy with [6], [7] we obtain the components of the vector $\Delta x_C'$ for system (11)-(13):

$$\Delta x_{\tau\rho}^k = \sum_{p=1}^{l} \nu_{t(\tau,\rho)^k,p} \widetilde{\beta}_p + \sum_{(i,j)\in U^*} \nu_{t(\tau,\rho)^k,l+\xi(i,j)} \widetilde{\beta}_{l+\xi(i,j)},$$

$$(\tau,\rho)^k \in U_C^k, k \in K, \ \widetilde{\beta} = \begin{pmatrix} \widetilde{\beta}_p, p = \overline{1,l} \\ \widetilde{\beta}_{\xi(i,j)}, (i,j) \in U_0 \end{pmatrix}. \quad (18)$$

The values of the components of the vectors $\widetilde{\beta}_p$ and $\widetilde{\beta}_{\xi(i,j)}$ are computed according to the following formulas:

$$\widetilde{\beta}_p = -\sum_{k\in K} \sum_{(\tau,\rho)^k \in U_N^k} R_p(L_{t(\tau,\rho)}^k) \Delta x_{\tau\rho}^k, \quad p = \overline{1,l}, \quad (19)$$

$$\widetilde{\beta}_{\xi(i,j)} = \Delta z_{ij} - \sum_{k\in K_0(i,j)} \sum_{(\tau,\rho)^k \in U_N^k} \delta_{\xi(i,j)}(L_{t(\tau,\rho)}^k) \Delta x_{\tau\rho}^k, \quad (i,j) \in U^*. \quad (20)$$

Taking into account the formula (14) we obtain:

$$\Delta\varphi(x) = \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_C^k} \widetilde{\Delta}_{\tau\rho}^k \Bigg[\sum_{p=1}^{l} \nu_{t(\tau,\rho)^k,p} \widetilde{\beta}_p$$

$$+ \sum_{(i,j)\in U^*} \nu_{t(\tau,\rho)^k,l+\xi(i,j)} \widetilde{\beta}_{l+\xi(i,j)} \Bigg] + \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_N^k} \widetilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k. \quad (21)$$

Let us introduce the following denotation:

$$r_p = \sum_{k\in K} \sum_{(\tau,\rho)^k} \widetilde{\Delta}_{\tau\rho}^k \nu_{t(\tau,\rho)^k,p}, \quad p = \overline{1,l},$$

$$r_{ij} = \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_q^k} \widetilde{\Delta}_{\tau\rho}^k \nu_{t(\tau,\rho)^k,l+\xi(i,j)}, \quad (i,j) \in U^*.$$

Taking into account the denotations made, we may represent $\Delta\varphi(x)$ the following way:

$$\Delta\varphi(x) = \sum_{(i,j)\in U^*} r_{ij} \Delta z_{ij} + \sum_{k\in K} \sum_{(\tau,\rho)^k \in U_N^k} \Big[\widetilde{\Delta}_{\tau\rho}^k$$

$$- \sum_{p=1}^{l} r_p R_p(L^k_{t(\tau,\rho)}) - \sum_{(i,j)\in U^*} r_{ij}\delta_{\xi(i,j)}(L^k_{t(\tau,\rho)}) \Bigg] \Delta x^k_{\tau\rho}$$

$$= \sum_{(i,j)\in U^*} \gamma_{ij}\Delta z_{ij} + \sum_{k\in K} \sum_{(\tau,\rho)^k \in U^k_N} \Delta^k_{\tau\rho}\Delta x^k_{\tau\rho}, \quad (22)$$

where Δz_{ij} is defined by formula (7),

$$\Delta^k_{\tau\rho} = \widetilde{\Delta}^k_{\tau\rho} - \sum_{p=1}^{l} r_p R_p(L^k_{t(\tau,\rho)}) - \sum_{(i,j)\in U^*} r_{ij}\delta_{\xi(i,j)}(L^k_{t(\tau,\rho)}),$$

$$(\tau,\rho)^k \in U^k_N, \quad k \in K, \quad \gamma_{ij} = r_{ij}. \quad (23)$$

3. Conditions of Optimality

Definition 1. A support plan $\{x, U_S\}$ is called nonsingular if the following conditions are met:

$$\begin{gathered} 0 < x^k_{ij} < d^k_{ij}, \quad k \in K^1_S(i,j), \quad (i,j) \in U, \\ x^k_{ij} > 0, \quad k \in K^0_S(i,j), \quad (i,j) \in U_0, \\ x^k_{ij} > 0, \quad k \in K_S(i,j)\backslash K^1_S(i,j), \quad (i,j) \in U\backslash U_0, \\ 0 < \sum_{k\in K_0(i,j)} x^k_{ij} < d^0_{ij}, \quad (i,j) \in U_0\backslash U^*. \end{gathered} \quad (24)$$

Theorem 1. *Let $\{x, U_S\}$ be a support plan. The following conditions are necessary for optimality of $\{x, U_S\}$ and are also sufficient if $\{x, U_S\}$ is nonsingular:*

$$\begin{array}{ll} x^k_{ij} = 0 & \text{if } \Delta^k_{ij} > 0, \\ x^k_{ij} = d^k_{ij} & \text{if } \Delta^k_{ij} < 0, \\ x^k_{ij} \in [0, d^k_{ij}] & \text{if } \Delta^k_{ij} = 0, k \in K^1_N(i,j), (i,j) \in U; \end{array} \quad (25)$$

$$\begin{array}{l} x^k_{ij} = 0 \text{ if } \Delta^k_{ij} > 0, \\ x^k_{ij} \geq 0 \text{ if } \Delta^k_{ij} = 0, k \in K^0_N(i,j), (i,j) \in U_0; \end{array} \quad (26)$$

$$\begin{array}{l} x^k_{ij} = 0 \text{ if } \Delta^k_{ij} > 0, \\ x^k_{ij} \geq 0 \text{ if } \Delta^k_{ij} = 0, k \in K_N(i,j)\backslash K^1_N(i,j), (i,j) \in U\backslash U_0; \end{array} \quad (27)$$

$$\begin{aligned} &\sum_{k\in K_0(i,j)} x_{ij}^k = 0 \qquad \text{if } \gamma_{ij} > 0,\\ &\sum_{k\in K_0(i,j)} x_{ij}^k = d_{ij}^0 \qquad \text{if } \gamma_{ij} < 0,\\ &\sum_{k\in K_0(i,j)} x_{ij}^k \in [0, d_{ij}^0] \ \text{if } \gamma_{ij} = 0, (i,j) \in U^*; \end{aligned} \tag{28}$$

Proof. The proof is given in [4]. □

In the criterion of an optimality (25)–(28) we used the analytical formula for computing reduced costs $\Delta_{\tau\rho}^k$:

$$\Delta_{\tau\rho}^k = \widetilde{\Delta}_{\tau\rho}^k - \sum_{p=1}^{l} r_p R_p(L_{t(\tau,\rho)}^k) - \sum_{(i,j)\in U^*} r_{ij}\delta_{\xi(i,j)}(L_{t(\tau,\rho)}^k),$$
$$(\tau,\rho)^k \in U_N^k, \quad k \in K, \quad \gamma_{ij} = r_{ij}.$$

For computing reduced costs $\Delta_{\tau\rho}^k$ we can build the vector $r = (r_p : p = \overline{1,l}; \gamma_{ij}, (i,j) \in U^*)$, $u_i = (u_i^k, k \in K(i))$, $i \in I$ as a solution of the potential system [2, 4].

We compute the reduced costs Δ_{ij}^k for the arcs $(i,j)^k \in U_N^k$, $U_N^k = U^k \backslash U_S^k$, $k \in K$ and for the arcs $(i,j)^k$, $k \in K_S^0(i,j)$, $(i,j) \in U^*$ using the following formula:

$$\Delta_{ij}^k = c_{ij}^k - \left(u_i^k - u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p \right). \tag{29}$$

One may check that the formulas (23) and (29) give the identical results for the problem (1)-(6). Strategy of application (23) or (29) are described in [1, 4, 6].

References

[1] Ravindra K. Ahuja, Thomas L. Magnanti, James B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, New Jersey (1993).

[2] R. Gabasov, F.M. Kirillova, O.I. Kostjukova, *Constructive Methods of Optimization. Part 3: Network Problems*, Minsk, BSU (1986), In Russian.

[3] L.A. Pilipchuk, Y.V. Malakhouskaya, D.R. Kincaid, M. Lai, Algorithms of solving large sparse underdetermined linear systems with embedded network structure, *East-West J. of Mathematics*, **4**, No. 2 (2002), 191-202.

[4] L.A. Pilipchuk, *Linear Inhomogeneous Problems of Networks Flows*, Minsk, BSU (2009), In Russian.

[5] L.A. Pilipchuk, A.S. Pilipchuk, Y.H. Pesheva, Algorithms for construction of optimal and suboptimal solutions in network optimization problem, *IJPAM*, **54**, No. 2 (2009), 193-205.

[6] L.A. Pilipchuk, I.V. Romanovski, Y.H. Pesheva, Inverse matrix updating in one inhomogeneous network flow programming problem, *Mathematica Balkanica*, New Series, **21**, No-s: 1-2 (2008), 329-338.

[7] L.A. Pilipchuk, E.S. Vecharynski, Y.H. Pesheva, Solution of large linear systems with embedded network structure for a non-homogeneous network flow programming problem, *Mathematica Balkanica*, New Series, **22**, No-s: 3-4 (2008), 233-252.