NON-COMMUTATIVE CURVES AND TILTING

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A *non-commutative curve* is a pair $(X, \mathcal{A}_X)$ i, where X is a (usual) algebraic curve and $\mathcal{A}_X$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. We denote by Coh$\mathcal{A}_X$ the category of coherent sheaves of $\mathcal{A}_X$-modules.

Let X be a (usual) projective curve over a field $\mathbf{k}$ such that all its components are rational and all singular points are simple double points, S be the set of singular points of X, $\pi : Y \to$ char↑∈∞ X be its normalization, $Y_1, Y_2, \dots, Y_s$ be the irreducible components of Y, all identified with the projective line $\mathbb{P}^1$, $\{x', x''\} = \pi^{-1}(x)$ for $x \in S$, $\mathcal{O}_X^* = \pi_* \mathcal{O}_Y$, $\mathcal{M} = \mathcal{O}_X \oplus \mathcal{O}_X^*$ and $\mathcal{A}_X = \mathcal{E}\backslash_{\mathcal{O}_X}(\mathcal{M})$.

Theorem 1. 1. gl.dim.$\mathcal{A}_X = 2$, *i.e.* $\mathrm{Ext}^2_{\mathcal{A}_X}$ *vanishes.*

2. $\mathcal{T} = \mathcal{O}_X^* \oplus \mathcal{O}_X^*(1) \oplus \bigoplus_{x \in S} \mathbf{k}(x)[-1]$, *where* $\mathbf{k}(x)$ *is the resudue field of the point* x, *is a tilting object in the derived category* $D^b(\mathrm{Coh}\mathcal{A}_X)$, *so* $D^b(\mathrm{Coh}\mathcal{A}_X) \simeq D^b(\mathbf{A}\text{-mod})$, *where* $\mathbf{A}_X = \mathrm{End}_{D^b(\mathrm{Coh}\mathcal{A}_X)}(\mathcal{T})$.

3. $\mathbf{A}_X \simeq \mathbf{k}\Gamma/I$, *where:*

Γ *is the quiver (oriented graph) with the set of vertices* $S \cup \{y_i, y_i' \mid 1 \leqslant i \leqslant s\}$ *and the set of arrows* $\{a_Z, b_Z \mid Z \in C\} \cup \{c_x', c_x'' \mid x \in S\}$ *such that* $a_i, b_i : y_i \to y_i'$, $c_x' : x \to y_j$ *and* $c_x'' : x_k \to y$ *if* $x' \in Y_j$, $x'' \in Y_k$;

I *is the ideal of the path algebra* $\mathbf{k}\Gamma$ *generated by the elements* $(\eta_i' a_j - \xi_i' b_j)c_x' = 0$ *and* $(\eta_i'' a_k - \xi_i'' b_k)c_x''$, *where* $(\xi_i' : \eta_i')$ *and* $(\xi_i'' : \eta_i'')$ *are, respectively, the homogeneous coordinates of* x_i' *on* Y_j *and of* x_i'' *on* Y_k.

Note that the algebra $\mathbf{A}_X$ first appeared in [1], where the authors showed that its representation type coincides with that of the curve X and asked for an *a priori* explanation of this phenomenon. Theorem 1 gives such an explanation, since the category Coh$\mathcal{O}_X$ naturally embeds into Coh$\mathcal{A}_X$.

We also present generalizations of Theorem 1 for other classes of curves (both usual and non-commutative).

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References

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MOTIVIC STRUCTURES IN NON-COMMUTATIVE GEOMETRY

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"Non-commutative geometry" is a term with many meanings. I am following the recent usage, which equates non-commutative geometry with "geometry of triangulated categories" — one attempts to treat a triangulated category, or maybe a DG or an A_∞-category, as a geometric object with all that it entails. The desire to do so comes originally from physics, where interesting A_∞-categories such as the Fukaya category appear in the context of topological field theories.

Mathematically, main motivation comes from the usual algebraic geometry, where one attempts to extract information about an algebraic variety X from its derived category $\mathcal{D}^b(X)$ of coherent sheaves. When passing from X to $\mathcal{D}^b(X)$, some information is certainly lost — different varieties can have equivalent derived categories. What is surprising is how much can be recovered. Here are some the invariants of X which are completely determined by $\mathcal{D}^b(X)$ (or rather, its A_∞-version):

1) algebraic K-theory $K^{\cdot}(X)$,

2) differential forms Ω^i_X — these correspond to Hochschild homology classes of $\mathcal{D}^b(X)$,

3) de Rham cohomology $H^{\cdot}_{DR}(X)$ — this corresponds to periodic cyclic homology of $\mathcal{D}^b(X)$.

It is expected that much more is true: loosely speaking, all the "motivic"structures which exits on the de Rham cohomology $H^{\cdot}_{DR}$ should also exist on the periodic cyclic homology of a nice enough A_∞-category $\mathcal{C}$.

In the talk, I will present a recent discovery in this direction: it turns out that for a nice enough A_∞-category $\mathcal{C}$ defined over a finite field k, or over the Witt vectors ring $W(k)$, the periodic cyclic homology $HP_{\cdot}(\mathcal{C})$ carries a natural action of the Frobenius map, and moreover, has a structure of a "filtered Diedonné module" of Fontaine-Lafaille — the p-adic analog of a mixed Hodge structure. This allows, among other things, to prove a Hodge-to-de Rham degeneration result for $\mathcal{C}$ using the well-known method of Deligne and Illusie. I will also discuss the relation with the notion of syntomic cohomology, and present a p-adic non-commutative version of the Beilinson conjecture and the regulator map.

QUANTUM LEIBNIZ ALGEBRAS AND THEIR CONFORMAL REPRESENTATIONS

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Quantum Lie algebras introduced by S. Woronowicz [1] are related to the axiomatic approach to the first order differential calculus (FODC) over a Hopf algebra. By definition, a quantum Lie algebra $\mathfrak{g}$ is a linear space equipped with a braiding $\sigma : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ and with a linear bracket product $\mu = [\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the following axioms:

$$\mu(\mathrm{id}\otimes\mu) = \mu(\mu\otimes\mathrm{id}) + \mu(\mathrm{id}\otimes\mu)(\sigma\otimes\mathrm{id}), \tag{1}$$

$$\sigma(\mathrm{id}\otimes\mu) - (\mu\otimes\mathrm{id})\sigma_{23}\sigma_{12} = 0, \tag{2}$$

$$\sigma(\mu\otimes\mathrm{id}) - (\mathrm{id}\otimes\mu)\sigma_{12}\sigma_{23} = (\mu\otimes\mathrm{id})(\mathrm{id}\otimes\sigma) - \sigma(\mathrm{id}\otimes\mu)(\sigma\otimes\mathrm{id}), \tag{3}$$

$$\mathrm{Ker}\,(\mathrm{id}-\sigma) \subseteq \mathrm{Ker}\,\mu, \tag{4}$$

If σ is the ordinary flip of tensor factors then (1) coincides with the Jacobi identity, (2) and (3) trivially hold, (4) represents skew-symmetry of μ.

Therefore, it is natural to call a system $(\mathfrak{g},\sigma,\mu)$ satisfying (1)–(3) to be a non-commutative analogue of a quantum Lie algebra, i.e., a *quantum Leibniz algebra*.

On the other hand, conformal algebras were introduced by V. Kac [2] as a tool of studying the structure and representation theory of vertex operator algebras. One of the most important