STATISTICAL ANALYSIS OF PARAMETER ESTIMATIONS OF SOME TIME SERIES MODELS

N.N. Troush, A.N. Yarosh
Belorussian State University, Minsk
e-mail: TroushNN@bsu.by, alexyar@yandex.ru

Abstract

This paper studies the M-estimation in a general conditionally heteroscedastic time series models. Sufficient conditions for strong consistency and asymptotic normality of the estimation are established.

1 Introduction

Conditionally heteroscedastic models are among most widely used models for financial time series. Aside from standard maximum likelihood method several other methods of parameter estimation for these models were proposed, such as quasi-maximum likelihood estimation [3], least deviation estimation [4] and B-estimation [2]. To generalize analysis of various methods in case of simple GARCH(p,q) model Berkes et al. in [1] used the concept of M-estimation and derived sufficient conditions for consistency and asymptotic normality of the estimation.
On the other hand, Straumann in [5] and Straumann and Mikosch in [6] proposed a stochastic recurrence equations (SRE) approach which allowed not to limit estimation analysis to special cases of the models, but their research was focused only on quasi-maximum likelihood method. In this work we combine the techniques from [1] and [5] to derive properties of general M- estimation with application to the whole class of conditionally heteroscedastic models. We show that the estimation is strongly consistent and asymptotically normal under mild condition for a model and likelihood function.

2 M-estimation

We study the M-estimation in a general conditionally heteroscedastic time series model in the following form:

$$\begin{cases} X_t = \sigma_t Z_t \\ \sigma_t = g_\theta\left(X_{t-1}, \sigma_{t-1}\right), \end{cases} \tag{1}$$

where the volatility process σ_t, $t \in \mathbb{Z}$ is nonnegative, Z_t, $t \in \mathbb{Z}$ is the sequence of i.i.d. random variables with $\mathrm{E}\left\{Z_0\right\} = 0$ and $\left\{g_\theta\left(X_t, \sigma_t\right) : \theta \in K \subset \mathbb{R}^d\right\}$ is a parametric family of nonnegative functions on $\mathbb{R} \times [0, \infty)$. We also require that σ_t is $\mathcal{F}_{t-1}$-measurable for every $t \in \mathbb{Z}$, where $\mathcal{F}_{t-1} = \sigma\left(Z_{t-1}, Z_{t-2}, \ldots\right)$.
Assume that θ belongs to some compact parameter space $K \subset \Theta \subset \mathbb{R}^d$ and that (1)

with true parameter $\theta = \theta_0$ has a unique stationary ergodic solution (X_t, σ_t), $t \in \mathbb{Z}$. For any initial value $c_0 \in [0, \infty)$ we define the following random function on K:

$$\hat{h}_t = \begin{cases} c_0 & t = 0 \\ \Phi_{t-1}(\hat{h}_t - 1) & t \geq 1, \end{cases} \tag{2}$$

where the random maps $\Phi_t : \mathbb{C}(K, [0, \infty)) \to \mathbb{C}(K, [0, \infty))$ are given by

$$[\Phi_t(s)](\theta) = g_\theta(X_t, s(\theta)), \ t \in \mathbb{Z}, \tag{3}$$

$\mathbb{C}(K, [0, \infty))$ denotes the set of continuous functions on K with values from $[0, \infty)$. We can regard that $\hat{h}_t$ is an estimate of the volatility σ_t under the parameter hypothesis θ which is based on the data $X_0, \ldots, X_t$.
Observe that $\hat{h}_t$ is a solution of the following SRE:

$$s_t = \Phi_{t-1}(s_{t-1}), \ t \in \mathbb{N}. \tag{4}$$

For establishing consistency of M-estimation we must find a stationary ergodic approximation h_t, $t \in \mathbb{Z}$, such that the error $\hat{h}_t - h_t$ converges to zero sufficiently fast as $t \to \infty$ and $h_t(\theta) = \sigma_t$ *a.s.* if and only if $\theta = \theta_0$. Straumann in [5] showed that the unique stationary solution of SRE (4) with index set $\mathbb{Z}$ provides the desired process h_t, $t \in \mathbb{Z}$.

Now we choose arbitrary positive function $f(x)$, $x \in \mathbb{R}$, and set

$$\hat{L}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \ln\left[\frac{1}{\hat{h}_t(\theta)} f\left(\frac{X_t}{\hat{h}_t(\theta)}\right)\right], \ \theta \in K. \tag{5}$$

The M-estimation is defined as the maximizer of $\hat{L}_n(\theta)$:

$$\hat{\theta}_n = \arg\max_{\theta \in K} \hat{L}_n(\theta). \tag{6}$$

Let $\phi(x, y) = \ln y f(xy)$ and $\phi_k(x, y) = \dfrac{\partial^k}{\partial y^k}\phi(x, y)$, $k = 1, 2, \ldots$, $y \in (0, \infty)$, $x \in \mathbb{R}$. We assume a set of conditions:

C1. Model (1) with $\theta = \theta_0$ admits a stationary ergodic solution (X_t, σ_t), $t \in \mathbb{Z}$ with $\mathrm{E}\{\ln \sigma_0\} < \infty$;

C2. h_t, $t \in \mathbb{Z}$ is a stationary ergodic process and

$$\left\|\hat{h}_t - h_t\right\|_K \xrightarrow[t \to \infty]{e.a.s.} 0,$$

where $\|\cdot\|_K$ denotes a uniform norm on K and $\xrightarrow{e.a.s.}$ denotes exponentially fast almost sure convergence;

C3. The class of functions $\{g_\theta|\theta \in K\}$ is uniformly bounded from below, that is

$$\exists \underline{g} > 0, \forall (x,s) \in \mathbb{R} \times [0,\infty), \forall \theta \in K, \; g_\theta(x,s) \geq \underline{g};$$

C4. The following identifiability condition holds for all θ on K:

$$h_0(\theta) = \sigma_0 \; a.s. \;\Leftrightarrow\; \theta = \theta_0;$$

C5. $\mathrm{E}\left\{\ln^+ \|h_0\|_K\right\} < \infty$;

C6. $\mathrm{E}\left\{\ln^+ |\phi_1(Z_0, y)|\right\} < \infty, \; \forall y > 0$;

C7. $\mathrm{E}\{\phi(Z_0, y)\} < \mathrm{E}\{\phi(Z_0, 1)\} < \infty, \; \forall y > 0$.

Theorem 1 *Under the conditions **C1-C7** the M-estimation* (6) *is strongly consistent:*

$$\hat{\theta}_n \xrightarrow[n\to\infty]{a.s.} \theta_0$$

.

Next we discuss the asymptotic normality of M-estimation. We formulate the following set of assumptions:

N1. Conditions **C1-C7** hold;

N2. For every $t \in \mathbb{Z}$ $\hat{h}_t(\theta)$, $h_t(\theta)$ are twice differentiable on K . $\mathrm{E}\left\{\ln^+ \|h'_0\|_K\right\} < \infty$, and

$$\left\|\hat{h}'_t - h'_t\right\|_K \xrightarrow[t\to\infty]{e.a.s.} 0, \; \left\|\hat{h}'_t - h'_t\right\|_K \xrightarrow[t\to\infty]{e.a.s.} 0,$$

N3. $\mathrm{E}\{|\phi_2(Z_0, y)|\} < \infty, \; \forall y > 0$;

N4. $\mathrm{E}\|l''_t\|_K < \infty$,

N5. $\mathrm{E}\{\phi_1^2(Z_0, 1)\} < \infty$;

N6. The components of the vector

$$\left.\frac{\partial g_\theta(X_0, \sigma_0)}{\partial \theta}\right|_{\theta=\theta_0}$$

are linearly independent.

N7. If $\mathrm{E}\left\{\dfrac{h'_0(\theta_0)}{\sigma_0}\right\} = 0$ holds, then $\mathrm{E}\{\phi_1(Z_0, 1)\} = 0$.

Theorem 2 *Under the conditions **N1-N7** the M-estimation* (6) *is strongly consistent and asymptotically normal:*

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \mathrm{V}),$$

where the asymptotic covariance matrix $\mathbf{V}$ *is given by*

$$\mathbf{V} = \frac{\mathrm{E}\{\phi_1^2(Z_0, 1)\}}{\left(\mathrm{E}\{\phi_2(Z_0, 1)\}\right)^2}\mathbf{A}^{-1}.$$

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