ESTIMATION OF THE LOCATION PARAMETER OF α-STABLE DISTRIBUTIONS

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Abstract

In this paper a method of estimating location parameter μ of α - stable distribution for $\alpha \in (0, 2]$ is provided. Firstly, the estimates of parameters α and σ with the characteristic function method (CF) are obtained, then the method of estimating of parameter μ based on the means of non-skewed transformations.

1 Introduction

In practice the estimation of parameters α - stable distributions is very important and now several estimating methods are known: maximum likelihood method [1,2], Hill method [3], quantile method [4], characteristic function method [5] etc. For estimating of parameters of symmetric distributions ($S\alpha S$) fractional lower order moment method, extreme value method, logarithmic moment method [6,7] and others are used. But now in the literature only a few methods of estimating location parameter μ exist. When $\alpha \to 1$ and $\beta \neq 0$, it is difficult to estimate μ. The estimation for parameter μ in the case of $\alpha \in (1, 2]$ was given in [8]. Estimation of parameter μ for the case $\alpha \in (0, 2]$, based on the property of stable distributions and transformations for α-stable random variables is given in this paper.

2 Method CF for estimation of parameters

From [9] we know that X is stable random variable if and only if the logarithm of its characteristic function $\varphi_X(t)$ has the following representation:

$$ln\varphi_x(t) = \begin{cases} -\sigma^\alpha|t|^\alpha(1 - i\beta sign(t)\tan(\frac{\pi\alpha}{2})) + i\mu t, \alpha \neq 1, \\ -\sigma|t|(1 + i\beta sign(t)\frac{2}{\pi}ln|t|) + i\mu t, \alpha = 1, \end{cases} \quad (1)$$

where $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma > 0, \mu \in R, t \in R$. We will use a designation $X \sim S_\alpha(\sigma,\beta,\mu)$.

The basic idea of CF method is to estimate characteristic function from the sample data and then estimate parameters α, σ and β, μ by using the real and imaginary parts of the logarithm of its characteristic function. However, since characteristic function of α - stable distributions is not continuous at $\alpha = 1$, it is difficult to obtain a good estimation of parameters β and μ.

3 Method of the estimation of the location parameter

On the basis of properties of stable distributions, we establish the following result.

Theorem 1 [10, 11]. *Let X_k be independent identically distributed stable random variables with parameters $\alpha, \sigma, \beta, \mu$, i.e. $X_k \sim S_\alpha(\sigma, \beta, \mu)$, and $a_k \in R$, $k = \overline{1,n}$, then*

$$Z = \sum_{k=1}^{n} a_k X_k \sim S_\alpha((\sum_{k=1}^{n} |a_k|^\alpha)^{1/\alpha}\sigma, \frac{\sum_{k=1}^{n} a_k^{(\alpha)}}{\sum_{k=1}^{n} |a_k|^\alpha}\beta, \sum_{k=1}^{n} a_k\mu),$$

where $a_k^{(\alpha)}=sign(a_k)|a_k|^\alpha$.

Three types of transformations of α-stable random variables can be derived from Theorem 1.

Lemma 1 *Let $X_k \sim S_\alpha(\sigma, \beta, \mu)$, $k = \overline{1,n}$ and $X_k^C \sim X_{3k} + X_{3k-1} - 2X_{3k-2}$, then*

$$X_k^C \sim S_\alpha([2+2^\alpha]^{1/\alpha}\sigma, [\frac{2-2^\alpha}{2+2^\alpha}]\beta, 0).$$

Lemma 2 *Let $X_k \sim S_\alpha(\sigma, \beta, \mu)$, $k = \overline{1,n}$ and $X_k^D \sim X_{3k} + X_{3k-1} - 2^{1/\alpha}X_{3k-2}$, then*

$$X_k^D \sim S_\alpha(4^{1/\alpha}\sigma, 0, [2-2^{1/\alpha}]\mu).$$

Lemma 3 *Let $X_k \sim S_\alpha(\sigma, \beta, \mu)$, $k = \overline{1,n}$ and $X_k^S \sim X_{2k} - X_{2k-1}$, then*

$$X_k^S \sim S_\alpha(2^{1/\alpha}\sigma, 0, 0).$$

Here X^C denotes centered transformation, X^D - non-skewed transformation, X^S - symmetrized transformation. Observe that the lengths of sequences X^C and X^D are 1/3 length of initial sequence, and length of sequence X^S - 1/2 length of initial sequence. Since the given in lemma 2 transformation can transform random sequence $X_k \sim S_\alpha(\sigma, \beta, \mu)$ to non-skewed distribution $X_k^D \sim S_{\alpha^D}(\sigma^D, \beta^D, \mu^D)$ and $\mu^D = median(X_k^D)$ [9], then,

$$\hat{\mu} = \hat{\mu}^D(2-2^{1/\alpha})^{-1}.$$

Theorem 2 *Estimation $\hat{\mu}$ of parameter μ is unbiased and consistent.*

4 Simulation

Considering that the length of sequence should be a multiple of 3, we simulate 9999 α - stable random variables and estimate location parameter μ for $\alpha \in [0.2, 1.8]$ (see figure 1).

Comparison of estimates of parameter μ by method in the paper and *CF* method in case of symmetric distribution $S_\alpha(1,0,0)$ with various values of parameter α is shown in figure 2.

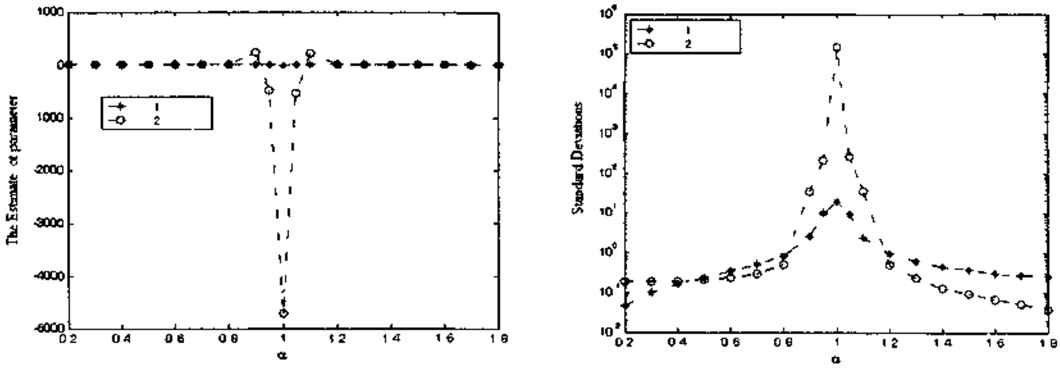

Figure 1: The estimate and standard deviations of estimation of parameter μ for $S_\alpha(2,0.9,1)$: 1-method in paper; 2-*CF* method

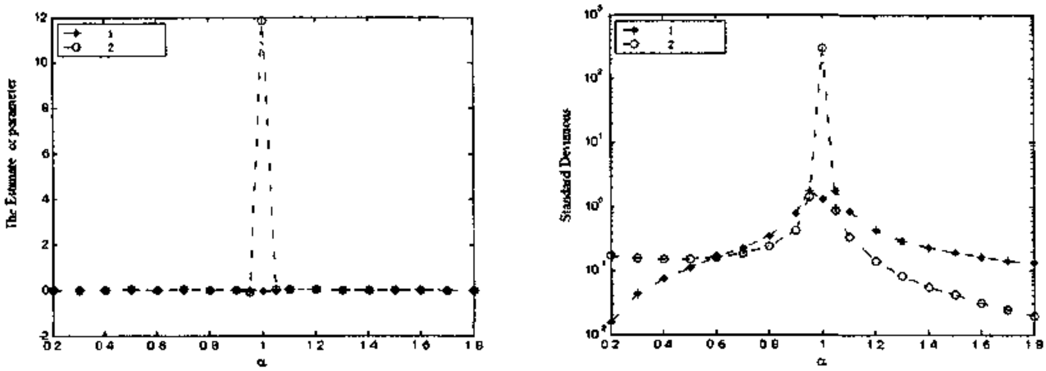

Figure 2: The estimate and standard deviations of estimation of parameter μ for $S_\alpha(1,0,0)$: 1-method in paper; 2-*CF* method

Table 1: Estimation of parameters μ with α close to 1

$S_\alpha(2,-0.9,1)(\mu=1)$					
α	0.90	0.95	1.00	1.05	1.10
Method in paper	1.2398 (2.4528)	2.0959 (9.7951)	-0.1201 (18.26)	0.4134 (9.2316)	0.8140 (2.3457)
CF Method	2.39×10^2 (32.503)	-4.78×10^2 (2.08×10^2)	-4.70×10^3 (1.489×10^5)	-5.30×10^2 (2.57×10^2)	2.25×10^2 (34.543)
$S_\alpha(1,0,0)(\mu=0)$					
α	0.90	0.95	1.00	1.05	1.10
Method in paper	1.0295×10^{-3} (0.7852)	-2.4514×10^{-2} (1.7413)	-3.7650×10^{-2} 1.3164)	-3.6197×10^{-2} (1.7552)	2.2179×10^{-2} (0.8410)
CF Method	3.5412×10^{-3} (0.4245)	-7.6779×10^{-2} (1.4505)	11.8660 (2.9999×10^2)	3.9201×10^{-2} (0.8615)	1.1582×10^{-2} (0.3355)

5 Conclusion

We can compare the estimation of parameter μ by the method offered in paper with the CF method at various α on the basis of the results obtained from two simulations of α - stable distributions which are shown table 1. The estimation of parameter μ for symmetric distribution $S_\alpha(1,0,0)$ by the CF method is better than the estimation for asymmetrical distribution. When estimating parameter μ with α close to 1, the method that proposed in article is better than the CF method. And for asymmetrical distribution $S_\alpha(2,-0.9,1)$ the method of estimating parameter μ proposed in paper is better than the CF method with α close to 1.

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