FUZZY EVALUATION OF THE RISK OF INVESTMENT IN SECURITIES IN THE PORTFOLIO OPTIMIZATION PROBLEM

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Investment in securities always associates with the risk, but in practice it is enough difficult to evaluate the risk quantitatively. There is no united opinion about quantitative evaluation of the risk in the theory of finance. Two models of the risk evaluation are the most popularized: Value-at-Risk (VaR) model and risk interpretation as standard deviation of the security return.

In VaR models maximum financial loss at a given level of confidence is used as the risk measure. Usually this measure is related with the certain security, but it may be considered for the entire portfolio [1]. There are three main groups of VaR models: historical simulation methods; Monte-Carlo simulation methods; variance-covariance models. VaR calculation is based on long-period historical statistic data processing that is unacceptable in our country because of absence, bad quality and non-comparability of the input data.

Optimal portfolio investment theory, proposed by H. Markowits, assumes that an expected security return may be represented as average value of return time series. The risk is represented as standard deviation of this time series [2].

Markowits's model assumes two different situations:

- return maximization at given risk level;
- risk minimization at given return level.

There are may be many different constraints in the optimization model (for example, the constraint of security number in the portfolio, the trading operations constraints, the constraints of the proportion "floor and ceiling", etc), but increasing of the constraints number leads to the complication of the optimal decision search process. To using this model it is necessary to know the security return value and its changes in previous time periods. However there is no confidence and there is no warranty that real and (or) nominal security return will change in future on the same regularities, as in earlier periods. In the contemporary economy conditions historical data analysis and some of the traditional probability theory methods are unacceptable because of non-stochastic nature of the economic information uncertainty.

In the article [3] an alternate way of the risk and return expression in the portfolio optimization model is described. If the long-period historical return values are unknown or could not to describe the return behavior in the present and in the future it is suggested to use expert evaluations and predictions of the return in the form of fuzzy trapezoidal numbers.

Fuzzy number $A = (a, b, \alpha, \beta)$ is called trapezoidal with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form [3]:

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha}, \text{if } a - \alpha \le t < a, \\ 1, \text{if } a \le t \le b, \\ 1 - \frac{t-b}{\beta}, \text{if } b < t \le b + \beta, \\ 0, \text{otherwise.} \end{cases} \tag{1}$$

Security return expression in the form of fuzzy trapezoidal number means that in expert's opinion the most possible security return value lays in $[a; b]$ interval, but return value may deviate from this interval into bad (negative) direction on value α or into good (positive) direction on value β.

If we denote the degree of expert's confidence in the return evaluation as γ, then γ-cut of the return may be represented as follows:

$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)] = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta], \forall\gamma \in [0, 1]. \tag{2}$$

Average value and standard deviation of a fuzzy index, according to article [3], are conformably calculated by the following formulae:

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma, \tag{3}$$

$$\sigma^2(A) = \frac{1}{2}\int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))^2 d\gamma. \tag{4}$$

For fuzzy trapezoidal number these parameters have the following values:

$$E(A) = \int_0^1 \gamma[a - (1-\gamma)\alpha + b + (1-\gamma)\beta]d\gamma = \frac{a+b}{2} + \frac{\beta - \alpha}{2}, \tag{5}$$

$$\sigma^2(A) = \frac{1}{2}\int_0^1 \gamma[a - (1-\gamma)\alpha + b + (1-\gamma)\beta]^2 d\gamma = \left[\frac{b-a}{2} + \frac{\alpha+\beta}{6}\right]^2 + \frac{(\alpha+\beta)^2}{72}. \tag{6}$$

Thus, the usage of fuzzy numbers for the return evaluation and prediction allows avoiding the necessity of historical data analysis which is unacceptable or impossible in the unstable economic conditions. Expected security return and risk level may be calculated on the base of expert evaluation of the return in the fuzzy number form.

In the article [3] values of average and standard deviation calculated by the formulas (5, 6) is used for evaluation of the portfolio utility (in the utility function). However in that work it is not taken into account that expert estimations not always may be correctly expressed as trapezoidal numbers. Furthermore, return representation in such form liquidating an uncertainty related with the absence of statistical data, generates other type uncertainty: it is supposed with the same probability that return may be equal to any value from $[a, b]$ interval. Meanwhile, it is very important for an investor to know the predicted value more exactly. In development of the suggested in the article [3] methodology we shall consider the particular cases of fuzzy expert evaluations of security returns in form of triangular numbers.

In general case the formulas (5, 6) describe fuzzy trapezoidal asymmetric number ($\alpha \neq \beta, \alpha \neq 0, \beta \neq 0, a \neq b$). In practice these conditions may be not observed. Such cases, their economic sense and statistical parameters (average value and standard deviation) are represented in table 1.

Table 1: Expert evaluations of securities return and their statistical parameters

Type of the number	Conditions	Economic sense	Average value	Standard deviation
Symmetric trapezoidal number	$\alpha = \beta, a \neq b, \alpha \neq 0, \beta \neq 0$	The most possible security return value lays in $[a; b]$ interval, but return value may deviate from this interval into bad or good direction on value α	$\frac{a+b}{2}$	$\left(\frac{b-a}{2} + \frac{\alpha}{3}\right)^2 + \frac{\alpha^2}{18}$
L-type trapezoidal number	$a \neq b, \alpha \neq 0, \beta = 0$	The most possible security return value lays in $[a; b]$ interval, but return value may deviate from this interval into bad direction on value α	$\frac{a+b}{2} - \frac{\alpha}{6}$	$\left(\frac{b-a}{2} + \frac{\alpha}{6}\right)^2 + \frac{\alpha^2}{72}$
R-type trapezoidal number	$a \neq b, \alpha = 0, \beta \neq 0$	The most possible security return value lays in $[a; b]$ interval, but return value may deviate from this interval into good direction on value β	$\frac{a+b}{2} + \frac{\beta}{6}$	$\left(\frac{b-a}{2} + \frac{\beta}{6}\right)^2 + \frac{\beta^2}{72}$
Asymmetric triangular number	$a = b, \alpha \neq \beta, \alpha \neq 0, \beta \neq 0$	The most possible security return value is equal to a, but return value may deviate from this value into bad direction on value α or into good direction on value β	$a + \frac{\beta-\alpha}{6}$	$\frac{(\alpha+\beta)^2}{24}$
Symmetric triangular number	$a = b, \alpha = \beta, \alpha \neq 0, \beta \neq 0$	The most possible security return value is equal to a, but return value may deviate from this value into bad or good direction on value α	a	$\frac{\alpha^2}{6}$
L-type triangular number	$a = b, \alpha \neq 0, \beta = 0$	The most possible security return value is equal to a, but return value may deviate from this value into bad direction on value α	$a - \frac{\alpha}{6}$	$\frac{\alpha^2}{24}$
R-type triangular number	$a = b, \alpha = 0, \beta \neq 0$	The most possible security return value is equal to a, but return value may deviate from this value into good direction on value β	$a + \frac{\beta}{6}$	$\frac{\beta^2}{24}$
Real interval	$a \neq b, \alpha = \beta = 0$	Security return value lays in $[a; b]$ interval	$\frac{a+b}{2}$	$\frac{(b-a)^2}{4}$
Real number	$a = b, \alpha = \beta = 0$	Security return value is equal to a	a	0

From this table we can see the main shortcoming of standard deviation usage as the risk measure: in this case it is not taken into account that positive deviation is not a risk but a positive tendency and positive deviation could not be minimized in the portfolio optimization problem. Meanwhile, positive return deviation increased its investment attractiveness and may lead to the rise of the market security price.

It is shown in the table 1 that standard deviation has the same values for the securities which returns has only positive or only negative predicted deviation, if the absolute values of these deviations are identical ($\frac{\alpha^2}{24} = \frac{\beta^2}{24}$ and $\left(\frac{b-a}{2} + \frac{\alpha}{6}\right)^2 + \frac{\alpha^2}{72} = \left(\frac{b-a}{2} + \frac{\beta}{6}\right)^2 + \frac{\beta^2}{72}$ if $\alpha = \beta$). This is a contradiction to the economic sense of the expert estimations.

It is recommended in case of using fuzzy expert evaluations of the security return:

- to use fuzzy triangular numbers with the notation $A = (a, \alpha, \beta)$ for the interpretation of expected return more exactly;
- to use $a + \frac{\beta-\alpha}{6}$ value for average value estimation;
- to use $\alpha - \beta$ value as the risk measure instead of standard deviation. This value was obtained by the following way. Positive mode deviation of return is equal to $\sigma^+(A) = \int_0^1 \gamma\left[(a + (1-\gamma)\beta) - a\right]d\gamma = \frac{\beta}{6}$, negative mode deviation is equal to $\sigma^-(A) = \int_0^1 \gamma\left[a - (a - (1-\gamma)\alpha)\right]d\gamma = \frac{\alpha}{6}$. Their difference is equal to $\frac{1}{6}(\alpha - \beta)$ and takes into account the compensation of the negative deviation by the positive one. A coefficient which equals $\frac{1}{6}$ may be not taking into account.

Thus, in the article portfolio optimization methodology with the fuzzy expert evaluations of security returns was analyzed and extended. It is shown that standard deviation using as risk measure not takes into consideration deviation directions (positive or negative) and that is why it describes portfolio optimization goal indelicately. Several recommendations about fuzzy return evaluations in the portfolio optimization model are suggested, which allow increasing of the accuracy of expert evaluations interpretation.

References

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