ON EVALUATION OF CONDITIONAL EXPECTATIONS OF SOME CLASS RANDOM FUNCTIONALS

A.D. Egorov
National Academy of Sciences, Institute of Mathematics
Minsk, BELARUS
e-mail: egorov@im.bas-net.by

Abstract

The exact formulas for conditional expectations of functionals defined on linear spaces with Gaussian and Poisson measures are received. Some examples are presented.

1 Introduction

Conditional expectations are used widely under investigations of sequential testing hypothesis problems. This paper is devoted to evaluation of conditional expectations of functionals defined on random variables taking their values in some linear spaces. Along with traditional approach the functional integral evaluation methods are used. Our formulas extend the results received in [2,3] for Winer integrals.

2 Gaussian variables

First consider evaluation of conditional mathematical expectation of functionals depending on random variables known as measurable linear functional in the space with Gaussian measure. Let γ is Gaussian measure on linear space X with zero mean value and covariance functional $K(\xi,\eta)$, $\xi,\eta \in X'$, X' is dual space to X. Denote H the closure of the set $\{\langle\xi,\cdot\rangle \ (\xi \in X')\}$ in the space $L_2(X,\gamma)$ and $\mathcal{H} \subset X$ dual to H; $\langle\xi,\cdot\rangle$ is continuous linear functional on X. It is known [4] that for $a \in \mathcal{H}$ and almost all $x \in X$ the next series is defined:

$$(a,x) = \sum_{k=1}^{\infty} \langle\varphi_k, x\rangle (a, e_k)_{\mathcal{H}},$$

where $\{\varphi_k\}, \{e_k\}$, $k = 1,2,\ldots$, are orthonormal bases in H and $\mathcal{H}$, respectively, with $\varphi_k \in X'$ for all k and $\langle\varphi_k, e_j\rangle = \delta_{kj}$; functionals (a,x) are called measurable linear functional. So we consider evaluation of conditional mathematical expectation $\mathbf{E}[(a,\cdot)|(c,\cdot) = \xi]$, where $(a,x),(c,x)\,(a,c \in \mathcal{H})$ are measurable linear functionals, $\xi \in \mathbb{R}$. We use the equality [1]

$$\mathbf{E}(Z|X)(\xi)\frac{dP_X}{dm}(\xi) = \lim_{\eta\to 0} \mathbf{E}[(\chi_{\xi,\eta} \circ X)Z],$$

where Z, X are random variables, $\chi_{\xi,\eta}(v)$ is characteristic function of the set $[\xi-\eta, \xi+\eta]$, $(dP_X/dm)(\xi)$ is the density of P_X with respect to Lebesgue measure m. The conditional expectation $\mathbf{E}(Z|X)(\xi)$ is defined to by any real valued $\mathcal{B}(R)$-measurable function on R such that

$$\int\limits_{X^{-1}(B)} Z(\omega)dP(\omega) = \int\limits_{B} \mathbf{E}(Z|X)(\xi)dP_X(\xi),$$

for $B \in \mathcal{B}(R)$, where $P_X(B) = P(X^{-1}(B))$.

Let $X = (c, x)$, $Z = (a, x)$. Applying the next formulas, which is reformulation in terms of mathematical expectations of corresponding formulas for functional integrals [4],

$$\mathbf{E}[g((a_1,\cdot),\dots,(a_n,\cdot))] = (2\pi)^{-n/2}[\det A]^{-1/2}\times$$
$$\times \int\limits_{R^n} g(u)\exp\Big\{-\frac{1}{2}(A^{-1}u,u)\Big\}d^n u,$$

where a_i, $i=\overline{1,n}$, - linear independent elements from $\mathcal{H}$; A is matrix with elements $a_{ij} = (a_i, a_j)_{\mathcal{H}_\gamma}$; $d^n u = du_1 \dots du_n$, we get

$$\mathbf{E}[(\chi_{\xi,\eta}\circ X)Z] = \mathbf{E}[\chi_{\xi,\eta}((c,x))(a,x)] = [(2\pi)^2\|a\|^2\|c\|^2 - 2(a,c)]^{1/2}\times$$
$$\times \int\limits_{\xi-\eta}^{\xi+\eta}\int\limits_{R} v\exp\Big\{-\frac{1}{2}\frac{u^2\|a\|^2}{\det A} - \frac{1}{2}\frac{v^2\|c\|^2}{\det A} + \frac{uv(a,c)}{\det A}\Big\}dudv,$$

where subscript $\mathcal{H}$ in norm and scalar product is omitted.

Evaluating multiple integral and going to limit under $\eta \to 0$, we will have

$$\mathbf{E}[(a,\cdot)|(c,\cdot)](\xi)\frac{dP_X}{dm}(\xi) = \xi\frac{(a,c)}{\|c\|^2}\frac{1}{\sqrt{2\pi\|c\|^2}}\exp\Big\{-\frac{\xi^2}{2\|c\|^2}\Big\}.$$

As

$$\frac{dP_X}{dm}(\xi) = \frac{1}{\sqrt{2\pi\|c\|^2}}\exp\Big\{-\frac{\xi^2}{2\|c\|^2}\Big\},$$

it follows from here that

$$\mathbf{E}[(a,\cdot)|(c,\cdot)=\xi] = \xi\frac{(a,c)}{\|c\|^2}.$$

Then go over consideration of mathematical expectations of the form:

$$\mathbf{E}\Big[e^{i(a,\cdot)}|(c,\cdot)\Big](z),$$

where $(a,\cdot), (c,\cdot)$ are measurable linear functionals on X, $z \in \mathbb{R}$.

By of tightness of exponentials in the space $L_2(X,\gamma)$, one can use exact formulas for evaluation their conditional expectations for approximate evaluation of conditional expectations of random functionals from this space.

To get desirable formula we use known relation [1]

$$\mathbf{E}(Y|X)(z)\frac{dP_X}{dm}(z) == \frac{1}{(2\pi)^k}\int\limits_{R^k} \exp^{-i(u,z)_{R^k}} \mathbf{E}\Big[e^{i(u,X)_{R^k}}Y\Big]m(du),$$

where Y is a real random variable with $\mathbf{E}[|Y|] < \infty$, X is k-dimensional random vector; the distribution P_X of random variable X absolutely continuous with respect to Lebesgue measure in R^k, $(dP_X/dm)(z)$ is the density of measure P_X with respect to measure m.

In our case $Y = e^{i(a,x)},\ X = (c,x),\ x \in X$. Using the above formula for calculation of functional integrals with integrands which are functions depending on measurable linear functionals we will have

$$\mathbf{E}\Big[e^{iu(c,\cdot)}e^{i(a,\cdot)}\Big] = \int\limits_X \exp\Big\{i\langle uc+a,x\rangle\Big\}d\gamma(x) = \exp\{-\frac{1}{2}\|uc+a\|^2\}.$$

Using explicit form of density

$$\frac{dP_X}{dm}(z) = \frac{1}{\sqrt{2\pi\|c\|^2}}\exp\Big\{-\frac{z^2}{2\|c\|^2}\Big\},$$

after some evaluations we get the formulas

$$\mathbb{E}\Big[e^{i(a,\cdot)}|(c,\cdot)\Big](z) = \exp\Big\{i\frac{z(a,c)}{\|c\|^2} - \frac{1}{2}\Big[\|a\|^2 - \frac{(a,c)^2}{\|c\|^2}\Big]\Big\}.$$

In particular, if $(a,x) = \langle\xi,x\rangle$ and $(c,x) = \langle\eta,x\rangle$, where $\xi,\eta \in X'$, we need to take $K(\xi,\eta)$ instead of (a,c) and $K(\cdot,\cdot)$ instead of $\|\cdot\|^2$.

One can see that conditional measure (under condition $\langle\eta,x\rangle = z$) is Gaussian measure with mathematical expectation $m_\eta(\xi) = zK(\xi,\eta)/K(\eta,\eta)$ and correlation functional $K_\eta(\xi,\xi) = \frac{1}{2}\Big[K(\xi,\xi) - (K^2(\xi,\eta)/K(\eta,\eta))\Big]$.

Received above formulas for conditional mathematical expectation it is not difficult extend to the case when $X = ((c_1,x),\ldots,(c_k,x))$, where $(c_1,x),\ldots,(c_k,x)$ are measurable linear functionals; $c_1,\ldots,c_k$ are orthogonal elements from $\mathcal{H}$:

$$\mathbb{E}\Big[e^{i(a,\cdot)}|(c_1,\cdot),\ldots,(c_k,\cdots)\Big](z_1,\ldots,z_k) =$$

$$= \exp\Big\{i\sum_{j=1}^k z_j\frac{(a,c_j)}{\|c_j\|^2} - \Big[\|a\|^2 - \sum_{j=1}^k\frac{(a,c_j)^2}{\|c_j\|^2}\Big]\Big\}.$$

3 Poisson variables

In this section random functionals are the measurable functionals defined on the space X of trajectories $x(\cdot)$ of Poisson process with measure μ_P. The measure is defined by characteristic functional

$$\chi_P(\xi) = \exp\Big\{\lambda \int_0^T \Big[e^{i\langle\eta,1_{[u,T]}\rangle} - 1 - \langle\eta, 1_{[u,T]}\rangle\Big] du\Big\},$$

where $1_{[u,T]}$ is characteristic function of interval $[u, T], \eta \in X'$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of [0,T]. Define the functions on [0,T] by equalities $[x](t) = \sum_{k=1}^{N}(x(t_k) - x(t_{k-1}))1_{[t_k,T]}(t), \quad [v](t) = \sum_{k=1}^{N} v_k 1_{[t_k,T]}(t), \ v = (v_1, \dots, v_N)$.

Let F is Borel measurable and integrable with respect to Poisson measure on X. Then the next formula is valid

$$\mathbf{E}[F(x(\cdot))|\, x(t_1) = v_1, \dots, x(t_N) = v_n] = \mathbf{E}(f(x - [x] + [v])).$$

The proof of the formula follows from

$$\int_{\Psi^{-1}(B)} F(x(\cdot))d\mu_P(x) = \int_B \mathbf{E}[F(x(\cdot) - [x](\cdot) + [v](\cdot))]dP_\Psi(v),$$

where the map $\Psi : X \to R^N$ is defined by $\Psi(x(\cdot)) = \{x(t_k) - x(t_{k-1}), k = 1, \dots, N\}$, $B \in \mathcal{B}(R^N)$, and uses the equality

$$\mathbf{E}[f(\nabla x(t_1), \dots, \nabla x(t_n))] = \sum_{k_1,\dots,k_n=0}^{\infty} f(k_1, \dots, k_n)P_\Psi(k_1, \dots, k_N),$$

where $P_\Psi(k_1, \dots, k_N) = \prod_{i=1}^{n} P_{\nabla t_i}(k_i), \quad P_{\nabla t_i}(k_i) = (\lambda\nabla t_i)^{k_i}/k_i!\, e^{-\lambda\nabla t_i}, \quad \nabla x(t_k) = x(t_k) - x(t_{k-1}), \ \nabla t_i = t_i - t_{i-1}$.

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