ASYMPTOTIC PROPERTIES OF MOMENTS OF MODIFIED PERIODOGRAM SMOOTHED BY SPECTRAL WINDOWS

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Abstract

Asymptotic properties of mathematical expectation of smoothed modified periodogram are investigated.

1 Introduction

The investigation of stable process properties with index α, $0 < \alpha < 2$, in time series by traditional methods seems to be difficult because they only have final moments of p-order, $0 < p < \alpha$. In this paper asymptotic properties of mathematical expectation of smoothed modified periodogram are investigated.

2 Asymptotic properties of mathematical expectation of smoothed modified periodogram

Consider a real symmetric stable stationary random process $X(t)$, $t \in Z = \{0, \pm 1, \pm 2, \ldots\}$, with discrete-time,a characteristic parameter α, $\alpha \in (0, 2]$, (see [1], [2], [3]) which has spectral representation

$$X(t) = \int_{\Pi} \cos(\lambda t) d\xi(\lambda), \tag{1}$$

where $t \in Z$, $\xi(\lambda)$ – is a real stable random process with independent increments and

$$\{E|d\xi(\lambda)|^p\}^{\alpha/p} = C(p, \alpha) f(\lambda) d\lambda, \tag{2}$$

for $0 < p < \alpha < 2$, the constant $C(p, a)$ depends only on p, α, and function $f(\lambda)$, $\lambda \in \Pi = [-\pi, \pi]$, called the spectral density of process $X(t)$, $t \in Z$, is nonnegative, even, integrable and periodic with period 2π.

Let the spectral density be unknown and it is necessary to construct a consistent estimate according to $2T+1$ observations – $X(-T), ..., X(0), ..., X(T)$. In the paper [2] the modified periodogram $I_T(\lambda)$, $\lambda \in \Pi$, has been investigated as a spectral density estimate. It has been shown that this statistic being an asymptotically unbiased estimate for the function $(f(\lambda))^{p/\alpha}$, $\lambda \in \Pi$, $0 < p < \alpha < 2$, is not consistent.

Consider the statistic [4] as the consistent estimate of spectral density

$$\hat{f}_T(\lambda) = \left(\tilde{f}_T(\lambda)\right)^{\alpha/p}, \tag{3}$$

where

$$\tilde{f}_T(\lambda) = \int_\Pi W_T(\lambda - u) I_T(u) du = \int_\Pi W_T(u) I_T(\lambda + u) du, \tag{4}$$

$0 < p < \alpha < 2$, $I_T(u)$, $u \in \Pi$, – a modified periodogram defined in papers [1], [3], the spectral window $W_T(\lambda)$, $\lambda \in \Pi$, is nonnegative, even, integrable and 2π -periodic function satisfying the conditions

$$1) \int_\Pi W_T(u) du = 1, T = 1, 2, ... \tag{5}$$

$$2) \lim_{T\to\infty} \int_{\Pi \setminus \{|u| \le \delta\}} W_T(u) du = 0, 0 < \delta < \pi. \tag{6}$$

Theorem. *If spectral density $f(\lambda)$, $\lambda \in \Pi$, positive, continuous in the point $\lambda_0 \in \Pi$ and bounded on Π, then for mathematical expectation of the statistic $\tilde{f}_T(\lambda)$,$\lambda \in \Pi$, defined by the equality (4) the following limit correlation is true*

$$\lim_{T\to\infty} E\tilde{f}_T(\lambda_0) = (f(\lambda_0))^{p/\alpha}, 0 < p < \alpha. \tag{7}$$

Proof. As is known from paper [1]

$$EI_T(\lambda) = (\gamma_T(\lambda))^{p/\alpha},$$

where

$$\gamma_T(\lambda) = \left(\frac{Q(\lambda)}{2}\right)^\alpha \int_\Pi |\ H_T(\lambda - u) + H_T(\lambda + u)|^\alpha f(u) du,$$

$$Q(\lambda) = \begin{cases} 2^{1-1/\alpha}, & \lambda \ne 0, \quad 0 < \alpha \le 1, \\ 2^{-1/\alpha}, & \lambda \ne 0, \quad 1 < \alpha < 2, \\ 1, & \lambda = 0. \end{cases}$$

Using the properties of mathematical expectation and the equality (4), we get

$$E\tilde{f}_T(\lambda_0) = \int_\Pi W_T(u) \left(\gamma_T(\lambda_0 + u)\right)^{p/\alpha} du.$$

Considering the property (5) of function $W_T(\lambda)$, we have

$$\left|E\tilde{f}_T(\lambda_0) - (f(\lambda_0))^{p/\alpha}\right| = \int_\Pi W_T(u) \left|(\gamma_T(\lambda_0 + u))^{p/\alpha} - (f(\lambda_0))^{p/\alpha}\right| du = J.$$

Using the inequality $|x^r - y^r| \le \frac{r}{2} |x - y| (x^{r-1} + y^{r-1})$, true for any $x, y > 0$ and $r \in (0, 1) \cup (2, +\infty)$, we receive

$$J \le \int_{\Pi} W_T(u) \frac{p}{2\alpha} \Big((\gamma_T(\lambda_0+u))^{p/\alpha-1} + (f(\lambda_0))^{p/\alpha-1} \Big) |\gamma_T(\lambda_0+u) - f(\lambda_0)| \le$$

$$\le \max_{u\in\Pi} \frac{p}{2\alpha} \Big((\gamma_T(\lambda_0+u))^{p/\alpha-1} + (f(\lambda_0))^{p/\alpha-1} \Big) \int_{\Pi} W_T(u) |\gamma_T(\lambda_0+u) - f(\lambda_0)| \; du.$$

Denote $C = \max_{u\in\Pi} \frac{p}{2\alpha} \Big((\gamma_T(\lambda_0+u))^{p/\alpha-1} + (f(\lambda_0))^{p/\alpha-1} \Big)$.

$$J \le C \int_{\Pi} W_T(u) |\gamma_T(\lambda_0+u) - f(\lambda_0+u) + f(\lambda_0+u) - f(\lambda_0)| \; du \le$$

$$\le C \int_{\Pi} W_T(u) |\gamma_T(\lambda_0+u) - f(\lambda_0+u)| \, du + C \int_{\Pi} W_T(u) |f(\lambda_0+u) - f(\lambda_0)| \, du.$$

Making use of the inequality $|\gamma_T(\lambda_0) - f(\lambda_0)| \le \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+v) - f(\lambda_0)| \, dv$, we find out that

$$J \le C \int_{\Pi} W_T(u) \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+u+v) - f(\lambda_0+v)| \, dvdu +$$

$$+ C \int_{\Pi} W_T(u) |f(\lambda_0+u) - f(\lambda_0)| \, du =$$

$$= C \int_{\Pi} W_T(u) \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+u+v) - f(\lambda_0) + f(\lambda_0) - f(\lambda_0+v)| \, dvdu +$$

$$+ C \int_{\Pi} W_T(u) |f(\lambda_0+u) - f(\lambda_0)| \, du \le$$

$$\le C \int_{\Pi} W_T(u) \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+u+v) - f(\lambda_0)| \, dvdu +$$

$$+ C \int_{\Pi} W_T(u) \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+v) - f(\lambda_0)| \, dvdu +$$

$$+ C \int_{\Pi} W_T(u) |f(\lambda_0+u) - f(\lambda_0)| \, du =$$

$$= C \int_{\Pi} W_T(u) \int_{\Pi} |H_T(v)|^{\alpha} |f(\lambda_0+u+v) - f(\lambda_0)| \, dvdu +$$

$$+ 2C \int_{\Pi} W_T(u) |f(\lambda_0+u) - f(\lambda_0)| \, du.$$

Due to the continuity of function $f(\lambda)$, $\lambda \in \Pi$, in the point $\lambda_0 \in \Pi$, for any $\varepsilon > 0$ there are such δ_1 and δ_2, that when $|u| \le \delta_1$, $|v| \le \delta_2$, the inequalities will be fulfilled

$$|f(\lambda_0+u+v) - f(\lambda_0)| \le \frac{\varepsilon}{2},$$

$$|f(\lambda_0+u) - f(\lambda_0)| \le \frac{\varepsilon}{2}.$$

Let us split the integration area in J in the following way

$$J \le C \int_{|u|\le\delta_1} W_T(u) \int_{|v|\le\delta_2} |H_T(v)|^\alpha |f(\lambda_0+u+v)-f(\lambda_0)|\, dvdu+$$

$$+C \int_{\Pi\setminus\{|u|\le\delta_1\}} W_T(u) \int_{|v|\le\delta_2} |H_T(v)|^\alpha |f(\lambda_0+u+v)-f(\lambda_0)|\, dvdu+$$

$$+C \int_{|u|\le\delta_1} W_T(u) \int_{\Pi\setminus\{|v|\le\delta_2\}} |H_T(v)|^\alpha |f(\lambda_0+u+v)-f(\lambda_0)|\, dvdu+$$

$$+C \int_{\Pi\setminus\{|u|\le\delta_1\}} W_T(u) \int_{\Pi\setminus\{|v|\le\delta_2\}} |H_T(v)|^\alpha |f(\lambda_0+u+v)-f(\lambda_0)|\, dvdu+$$

$$+2C \int_{|u|\le\delta_1} W_T(u) |f(\lambda_0+u)-f(\lambda_0)|\, du+$$

$$+2C \int_{\Pi\setminus\{|u|\le\delta_1\}} W_T(u) |f(\lambda_0+u)-f(\lambda_0)|\, du \le$$

$$\le C \int_{|u|\le\delta_1} W_T(u) \int_{|v|\le\delta_2} |H_T(v)|^\alpha |f(\lambda_0+u+v)-f(\lambda_0)|\, dvdu+$$

$$+C \max_{u,v\in\Pi} |f(\lambda_0+u+v)-f(\lambda_0)| \int_{\Pi\setminus\{|u|\le\delta_1\}} W_T(u) \int_{|v|\le\delta_2} |H_T(v)|^\alpha\, dvdu+$$

$$+C \max_{u,v\in\Pi} |f(\lambda_0+u+v)-f(\lambda_0)| \int_{|u|\le\delta_1} W_T(u) \int_{\Pi\setminus\{|v|\le\delta_2\}} |H_T(v)|^\alpha\, dvdu+$$

$$+2C \int_{|u|\le\delta_1} W_T(u) |f(\lambda_0+u)-f(\lambda_0)|\, du+$$

$$+2C \max_{u\in\Pi} |f(\lambda_0+u)-f(\lambda_0)| \int_{\Pi\setminus\{|u|\le\delta_1\}} W_T(u)du = J_1+J_2+J_3+J_4+J_5.$$

Thus taking into account in J_2 and J_5 the property of the spectral window (6), in J_3 the property of kernel function $|H_T(\lambda)|^\alpha$, $\lambda \in \Pi$, and in J_1 and J_4 using the continuity of the function $f(\lambda)$, $\lambda \in \Pi$, in the point $\lambda_0 \in \Pi$, the equality (7) takes place. ☐

References

[1] Soboleva T.V. (2001)The calculation of first two moments of enlarged periodogram of a discrete real stable stationary random process. *Vestnik BSU*, Vol. **2**, pp. 72-74.

[2] Soboleva T.V. (2002)The asymtotics of moments of enlarged periodogram of a discrete real stable process. *Mathematical methods in finance and econometrics*, pp. 195 - 200.

[3] Troush N.N. (1999). *Asymptotic methods of statistic analysis of time series*. BSU, Minsk.

[4] Masry E., Cambanis S. (1984)Spectral density estimation for stationary stable processes. *Stochastic Processes and Their Applications*.Vol. **18**, pp.1-31.