ON GENERALIZATIONS OF INEQUALITIES OF CHERNOFF-TYPE

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Abstract

In the work are obtained generalizations of the inequality proved by H. Chernoff for bound on the variance of an absolutely continuous function of a standard normal random variable.

Let X be a standard normal random variable (r.v.). H. Chernoff in [1] proved an inequality playing important role in the theory of statistical inferences:

for any real valued absolutely continuous function $g(x)$,

$$Dg(X) \leq \mathbf{E}(g'(X))^2. \tag{1}$$

It should be noted that the mentioned Chernoff inequality is exact since one can easy check that this inequality becomes the equality for linear functions $g(x)$.

A.A. Borovkov and S.A. Utev in [2] obtained an inequality essentially generalized inequality (1), namely, they proved an inequality of type (1) for an arbitrary r.v. with the distribution function having an absolutely continuous component.

Let ξ be a r.v. with the distribution function

$$F_\xi(x) = \alpha F_1(x) + (1-\alpha)F_2(x) \tag{2}$$

where $0 \leq \alpha \leq 1$, $F_1(x)$ have the probability density $f_1(x)$.

Suppose $F_\xi(x)$ satisfies the conditions:

$$\begin{aligned} \int_u^\infty x dF_\xi(x) &\leq cf_1(u) \quad \text{for } \ u \geq 0, \\ -\int_{-\infty}^u x dF_\xi(x) &\leq cf_1(u) \quad \text{for } \ u < 0 \end{aligned} \tag{3}$$

at some $c > 0$.

In [2], it is given the simple proof of the following

Theorem 1. *Let $F_\xi(x)$ satisfy conditions (2) and (3). Then for any absolutely continuous function $g(\cdot)$,*

$$Dg(\xi) \leq \frac{c}{\alpha} E\left(g'(\xi)\right)^2. \tag{4}$$

Remark 1. In the case of

$$\mathbf{P}(\xi < x) = F_\xi(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/1} du, \tag{5}$$

conditions (2) and (3) are realized at $\alpha = 1$, $c = 1$, and

$$f_1(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

To make sure of validity of the last assertion, it is sufficient to differentiate the equality

$$\int\limits_{x}^{\infty} u f(u) du = f(x).$$

Thus, inequality (4) generalizes the Chernoff inequality (1) sufficiently.

In the following theorem, we give generalization of inequality of Chernoff-type (4).

Theorem 2. *Let ξ and η be independent r.v.'s and $F_\xi(x)$ satisfy conditions (2) and (3). Then for any absolutely continuous function g(x) with $g(0) = 0$,*

$$Dg(\xi\eta) \leq \frac{c}{\alpha} \mathbf{E}\left[\eta^2 \left(g'(\xi\eta)\right)^2\right]. \tag{6}$$

In the case of $\mathbf{P}(\eta = 1) = 1$, inequality (6) implies estimation (4).

Remark 2. B.L.S. Prakasa Rao in [3] proved inequality (6) in the case of ξ is a standard normal r.v. (i.e. with distribution function (5)). Denote also that in [3], using a characterization of the normal distribution obtained by Ch. Stein in [4], lower bounds for $\mathbf{E}\left[g(\xi\eta)\right]^2$ are determined.

Further suppose that considered r.v.'s are defined in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The following theorem generalizes inequality (6).

Theorem 3. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent r.v.s' with a common distribution function $F(x)$. Let $F(x)$ satisfy conditions (2) and (3). Let $\mathcal{F}_i$ be σ-algebras generated by r.v.'s $\xi_1, \xi_2, \ldots, \xi_i$ for $1 \leq i \leq n$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). Suppose Y_i and T_i are $\mathcal{F}_{i-1}$-measurable and r.v.'s Y_j and T_j, $i \leq j \leq n$ are independent of ξ_i for $i \geq 1$. Then for any partially differentiable functions $g(\cdot, \ldots, \cdot)$ and $h(\cdot, \ldots, \cdot)$ from $\mathbb{R}^n$*

$$\left|Cov\left(g(\xi_1 Y_1, \ldots, \xi_n Y_n), h(\xi_1 T_1, \ldots, \xi_n T_n)\right)\right| \leq \sum_{i=1}^{n} \left(\mathbf{E}\left[Y_i \frac{\partial g}{\partial x_i}\right]^2 \mathbf{E}\left[Y_i \frac{\partial h}{\partial x_i}\right]^2\right)^{1/2}. \tag{7}$$

Remark 3. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent r.v.'s with common standard normal distribution function (5). Suppose that random vectors $(Y_1, \ldots, Y_i)$ and $(T_1, \ldots, T_i)$ are independent of $(\xi_1, \xi_{i+1}, \ldots, \xi_n)$ at $1 \leq i \leq n$. Then (7) holds.

The following results can be obtained as corollaries to Theorem 3.

Corollary 1. *Let X be a standard normal r.v., $g(\cdot)$ and $h(\cdot)$ be real valued absolutely continuous functions. Then*

$$|Cov(g(X), h(X))| \leq \left(\mathbf{E} \left[\frac{dg}{dX} \right]^2 \cdot \mathbf{E} \left[\frac{dh}{dX} \right]^2 \right)^{1/2} .$$

Corollary 2. *Let X_1, X_2, $\ldots$, X_n be independent r.v.'s with common distribution function (5). Further suppose that functions $g(\cdot, \ldots, \cdot)$ and $h(\cdot, \ldots, \cdot)$ from $\mathbb{R}^n$ have partial derivatives of the order 1. Then*

$$|Cov[g(X), h(X)]| \leq \sum_{i=1}^{n} \left(\mathbf{E} \left[\frac{\partial g}{\partial X_i} \right]^2 \mathbf{E} \left[\frac{\partial h}{\partial X_i} \right]^2 \right)^{1/2}$$

here $X = (X_1, \ldots . X_n)$.

Remark 4. If one passes to the limit in inequalities (6), (7), then he can obtain an analog of the inequality of Chernoff-type for stochastic integrals

$$\int_0^T \alpha(t) dw(t)$$

where a nonrandom function $\alpha(t) \in L_2(0, T)$, $w(t)$ is the standard Wiener process determined on $[0, T]$.

For example, Theorem 2.2 of [3] implies that for any absolutely continuous function $g(x)$,

$$D \left[g \left(\int_0^T \alpha(t) dw(t) \right) \right] \leq \int_0^T \alpha^2(t) dw(t) \mathbf{E} \left[g' \left(\int_0^T \alpha(t) dw(t) \right) \right]^2 .$$

Note also that the last inequality can be used for characterization of the Wiener process in the class of random processes with independent increments.

References

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