A Branch and Bound Method for Optimization Problems with Fuzzy Numbers

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Abstract: *The algorithm of branch and bound method for the minimization problem in the fuzzy setting is proposed. The use of branch and bound method for combinatorial transportation problem on permutations with fuzzy data is shown.*

***Keywords*:** fussy sets, combinatorial optimization.

1. INTRODUCTION

Do not taking into consideration that the needs of practice still exist combinatorial optimization problems on fuzzy sets have not found enough researching yet. In fact there are no methods for such problems, which take into account their characteristic properties. Therefore it is actual to consider such problems and methods of their investigation. In the paper the general approach for solving the problems of minimization in a fuzzy formulation, which is illustrated by a combinatorial transportation problem on permutations with fuzzy data, within the scope of branch and bound method (MBB) is substantiated

2. FORMULATION OF A PROBLEM

Let us have a function $F(x)$, which is set on a set X ($x \in X$) – fuzzy numbers; $F(x) \in X$, scilicet the value, which the function takes on, is a fuzzy set as well. Let there be given $D \subset X$ – the set of feasible fuzzy numbers.

Using operations [1] (including finding of minimum and maximum), an optimization problem on fuzzy numbers set can be formulates so: find

$$\min_{x \in D} F(x). \qquad (1)$$

3. A BRANCH AND BOUND METHOD IN THE OPTIMIZATION ON FUZZY SETS

Let us denote S – a certain list (array), n_{rec} – the variable, which has a sense of the number revised by the method of a feasible solution. Algorithm MBB for (1) is stated in next steps.

0. $S = \varnothing$; $n_{rec} = 0$. Set a feasible region D ($D \neq \varnothing$), and an objective function F on D.

1. The set D is broken up on subsets $D_1, \dots, D_n$ with properties: $D_i \neq \varnothing$; $D_i \bigcap D_j = \varnothing \quad \forall i, j \in J_n = \{1, 2, \dots, n\}$, $D = D_1 \bigcup \dots \bigcup D_n$. It is considered that sets $D_1, \dots, D_n$ are unramified and are not cut off. Let us call such a set «a bud», and properties of such sets – «properties of buds».

Let us assign the estimation $\nu_i(D_i) = \nu_i \in X$ – fuzzy number with the property $\nu_i \prec F(x) \quad \forall x \in D_i$ each set, which does not belong to S and which is a bud, where a sign $\prec$ – is the sign of linear order on the set of fuzzy numbers X [1]. Let us add them in the list S of buds with estimations. Name n for the number of buds $|S|$.

2. The check: $S = \varnothing$? If «yes» – go to the step 16. If «no» – go to the step 3.

3. Chose an arbitrary bud D_i.

4. The check: if the number of elements $|D_i|$ in the set D_i is equal to one: $|D_i| = 1$? If «yes» – go to the step 6. If «no» – go to the step 5.

5. Have $|D_i| \neq 1$ (more precisely, it means $|D_i| > 1$), break up (branch out) D_i as D, have passed to the step 1.

6. Let us name a single-element bud as «leaf», so $D_i = \{x_{n_{rec}}\}$, $x_{n_{rec}} \in D$. The leaf D_i excludes from S. Let's compute $F_{n_{rec}} = F(x_{n_{rec}})$, using operations [1] for fuzzy numbers.

7. Check: $n_{rec} > 0$? If «no» – (i.e. $n_{rec} = 0$), then the pass to the step 8. Otherwise (i.e. $n_{rec} > 0$) – to the step 14.

8. Let us apply the point which give the record value of objective function the point x_0; $n_{rec} = 1$.

9. Set $i = 1$ (organize the beginning of the loop of the exhaustion of buds).

10. The check: $\nu_i \prec F_0$? If «yes» – go to the step 12, if «no» – go to the step 11.

11. Exclude the bud D_i from the list S. (Note that n is not changed during it, it is changed only on the step 1). It means the bud D_i cutting off.

12. Increase by one i. So $i := i + 1$.

13. The check: $i > n$? If «yes» – go to the step 2. If «no» – go to the step 10.

14. The check: $F_{n_{rec}} \succ F_0$? If «yes» – go to the step 2. If «no» – go to the step 15.

15. Assign to the record of objective function F_0 the value $F_{n_{rec}}$, so $F_0 := F_{n_{rec}}$, then $x_{rec} := x_{n_{rec}}$; $n_{rec} = n_{rec} + 1$. Go to the step 9.

16. The output of the result: the minimal value F_0 of objective function and the point x_{rec}, that gives it. The stop.

The algorithm flowchart is presented in the fig. 1-2.

The remark. This algorithm is a branch and bound algorithm in general (it means for unfuzzy and fuzzy formulation of a problem (1)), if a linear order $\prec$ is defined in the set of values of the objective function (in this connection in real numbers in the such specifications: it is $\geq$ on the step 14; it is $<$ on the step 10).

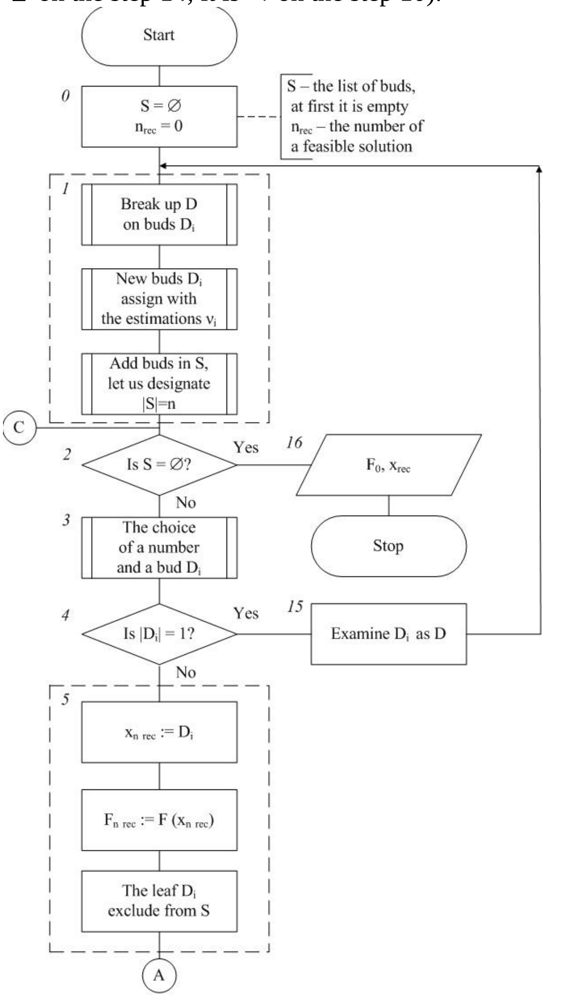

Fig.1 – Flowchart of the branch-and-bound method for minimization on the set of the fuzzy numbers.

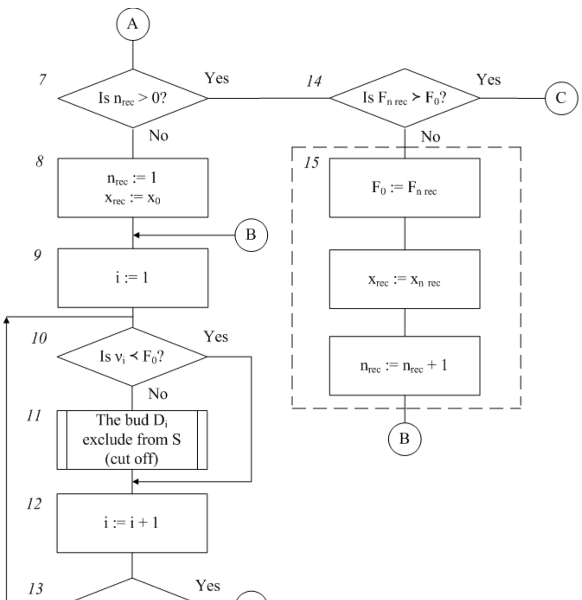

Fig.2 –Continuation of the flowchart of the branch-and-bound method for minimization on the set of the fuzzy numbers.

The way of branching of a feasible set (the step 1 (breaking up D on buds D_i); the step 3 – the choice of D_i) and an estimation D_i (defining the estimation on the step 1) essentially effect on the efficiency of a branch and bound method. By virtue of generality of the problem there are no recipes that act effectively in these cases. The methods of branching, cutting off, estimating are defined by the specification of the class of problems, which are considered. Cutting off occurs, as we can see from the algorithm, according to the classical condition for MBB: if $\nu_i(D_i) \prec F_0$ is not executed, then D_i – is cut off.

4. SOLVING OF A COMBINATORIAL TRANSPORTATION PROBLEM ON PERMUTATIONS IN FUZZY FORMULATION BY BRANCH AND BOUND METHOD

As it is known [1, 2], a combinatorial transportation problem on permutations (CTPP) in fuzzy formulation has the form: find the minimum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{2}$$

under conditions

$$\sum_{j=1}^{n} x_{ij} \prec a_i \quad \forall i \in J_m , \tag{3}$$

$$x = (x_{11},\ldots,x_{mn}) \in E_k(G) , \tag{5}$$

where c_{ij}, a_i, b_j, x_{ij} – fuzzy numbers [1], their minimum, sum, product, linear order $\prec$ are defined in [1], m, n, k – natural constants, and $E_k(G)$ – the set of fuzzy permutation [1], $m \cdot n = k$, $G = \{g_1,\ldots,g_k\}$ – multiset possible bodies of transportations, $J_m = \{1,2,\ldots,m\}$ – the set of the first m natural numbers.

Let us note that according to an economical content of the problem elements of the bearer of numbers c_{ij}, a_i, b_j, g_t are positive.

Let us consider the way of breaking up D on buds D_i. Arrange tariffs c_{ij}, $\forall i \in J_m$, $\forall j \in J_m$:

$$c_{i_1\tau_1} \succ c_{i_2\tau_2} \succ \ldots \succ c_{i_k\tau_k} ; \tag{6}$$

redesignated them so

$$c_1^* \succ c_2^* \succ \ldots \succ c_k^* . \tag{7}$$

Here i_l – it is the number of a row, τ_l – the number of a column, in which the $c_{i_l j_l} = c_l^*$ stand, which in the arrangement (7) stand in l-th position.

Let us have the bodies of transportation g_j, which constitute a multiset $G = \{g_1,\ldots,g_k\}$, are numbers according to an order:

$$g_1 \prec g_2 \prec \ldots \prec g_k . \tag{8}$$

We will redesignate it for convenience of the next exposition so: $g_1^0 \prec g_2^0 \prec \ldots \prec g_k^0$, $G^0 = G$.

We form buds is in such order. The set D is broken up on subsets according to an order:

D_1 – is subset D, where $x_{i_1 j_1} = g_k^0$;

D_2: $x_{i_1 j_1} = g_{k-1}^0$;

…

$D_i:\ x_{i_1 j_1} = g^0_{k-i+1}$;

…

$D_k:\ x_{i_1 j_1} = g^0_1$.

We choose the number i of buds the first level $D_1,\dots,D_k$ on the step 3 MBB by turns from 1-st until k-th.

If the bud of the first level $D_i = D_{\tau_1}$ is chosen, then it is considered as D on the step 5. It means the next. The difference of the multiset G and $\{g_{k-i+1}\}$ is formed and is used in the capacity of $G^1 = G^0 - \{g^0_{k-\tau_1+1}\}$, where $G^1 = \{g^1_1,\dots,g^1_{k-1}\}$ and elements are numbered so: $g^1_1 \prec \dots \prec g^1_{k-1}$. Buds of the second level are formed in such order:

$(D_{\tau_1})_1 = D_{\tau_1 1}$ – is the subset of the set D_{τ_1}, in which $x_{i_2 j_2} = g^1_{k-1}$.

$(D_{\tau_1})_2 = D_{\tau_1 2}$ – is the subset of the set D_{τ_1}, in which $x_{i_2 j_2} = g^1_{k-2}$.

…

$D_{\tau_1\tau_2}$ – is the subset of the set D_{τ_1}, in which $x_{i_2 j_2} = g^1_{k-\tau_2}$.

…

$D_{\tau_1(k-1)}:\ x_{i_2 j_2} = g^1_1$.

We choose the number τ_2 of buds the second level $D_{\tau_1\tau_2}$ on the step 3 of MBB by turns from $\tau_2 = 1$ until $\tau_2 = k-1$.

If on the level 1 the bud of r-th level $D_{\tau_1\tau_2\dots\tau_r}$ is chosen, which on the step 5 is considered as D, then it means the next.

The difference G^r of multisets G^{r-1} and $\{g^{r-1}_{k-\tau+1}\}$ is formed, that $G^r = G^{r-1} - \{g^{r-1}_{k-\tau+1}\}$, where $G^r = \{g^r_1,\dots,g^r_{k-r}\}$ and elements G^τ are numbered so:

$$g^r_1 \prec \dots \prec g^r_{k-r}.$$

Buds of the ($r+1$)-th level are formed in such order:

$(D_{\tau_1\tau_2\dots\tau_r})_1 = D_{\tau_1\tau_2\dots\tau_r 1}$ – is the subset $D_{\tau_1\dots\tau_r}$, when $x_{i_{\tau_r} j_{\tau_r}} = g^r_{k-r}$;

$(D_{\tau_1\tau_2\dots\tau_r})_2 = D_{\tau_1\tau_2\dots\tau_r 2}:\ x_{i_{\tau_r} j_{\tau_r}} = g^r_{k-r-1}$;

…

$D_{\tau_1\tau_2\dots\tau_r\tau_{r+1}}:\ x_{i_{\tau_r} j_{\tau_r}} = g^r_{k-r-\tau_{r+1}+1}$;

…

$D_{\tau_1\tau_2\dots\tau_r(k-r)}:\ x_{i_{\tau_r} j_{\tau_r}} = g^r_1$.

It is clear, that levels of buds are no more k.

It is clear, that certain buds will be empty, as the one from limitations in (3) can not be executed.

The remark. We can use any linear order on the set of tariffs of volumes of transportation instead of (6)-(8).

The remark. It is easy to see that for fuzzy numbers with a discrete bearer [1] and a sum [1] the empty bud can be cut off. That, if $(a+b) \prec (a+b)+c$, that, if $d = \{(d_1 \mid \mu_1),\dots,(d_t \mid \mu_t)\}$ $\forall i \in J_t$ $d_i \ge 0$, then

$$(a+b) \prec (a+b)+c+d. \qquad (9)$$

In other words, if on the certain level the condition (3) for a bud was not executed, it means that it would not execute for buds, which are formed by the partition this bud (on next levels).

5. ESTIMATING OF FEASIBLE SUBSETS IN THE BRANCH AND BOUND METHOD

Let us describe, as the estimation of the subset $D_{\tau_1\tau_2\dots\tau_r}$ – of the bud of r-th level is computed.

When input data are real numbers, the property of the estimation $\xi(D)$ of subset D, which provides the work of BBM, are so: if $D_i \subset D$, then $\xi(D_i) \ge \xi(D)$.

The following statement follows from this property.

The theorem 1. If $|D_{i_n}| = 1$, and the functional ξ, which is given on sets $D_{i_1},\dots,D_{i_n}$ is such, that $\xi(D_{i_n}) = \xi(x) = F(x)$, $\xi(D_{i_j}) \prec \xi(D_{i_{j+1}})$, $\forall j \in J_{n-1}$, then the value of the functional $\xi(D)$ can be the estimation of a feasible subset in BBM.

The proof. From the fact that $\xi(D_{i_j}) \prec \xi(D_{i_{j+1}})$ $\forall j \in J_{n-1}$ and $\xi(D_{i_n}) = \xi(x) = F(x)$ follows that $\xi(D_{i_j}) \prec F(x)$ $\forall x \in D_{i_j}$ $\forall j \in J_n$. As has to be proved.

The theorem 2. If $c_\tau = \{(c^\tau_1 \mid \mu^{c_\tau}_1),\dots,(c^\tau_k \mid \mu^{c_\tau}_k)\}$, $g_t = \{(g^t_1 \mid \mu^{g_t}_1),\dots,(g^t_l \mid \mu^{g_t}_l)\}$, where $c^\tau_i \ge 0$ $\forall i \in J_n$; $g^t_j \ge 0$ $\forall j \in J_l$, then the number $H(C)$ can serve the estimation $\xi(D)$ of the set D in BBM. $H(C)$ is characteristic comparator of fuzzy number C, where

$$C = \sum_{x \in G^B} c_j x^B_j \qquad (10)$$

– values of the part of the objective function, so those its summands, in which the variables are defined.

The proof. The proof проведемо for fuzzy numbers with a discrete bearer. Let a, b are fuzzy numbers with a discrete bearer. If $a \prec b$, then $H(a) \le H(b)$ (see the statement 2.7 of the part 2 from [1]).

If $a \prec b$, then $a \prec b+c$, if $c = \{(c_1 \mid \mu_1),\dots,(c_k \mid \mu_k)\}$ and $c_i \ge 0$ $\forall i \in J_k$ (see the theorem 2.8 of the part 2 from [1]).

Let us prove, that the characteristic comparator $H(C)$ can be taken as an estimation.

The subset of the next level of BBM gives one more summand in the defined part of objective function – let $\Delta C = c_\tau g_t$ (note, that $c_\tau = \{(c^\tau_1 \mid \mu^{c_\tau}_1),\dots,(c^\tau_k \mid \mu^{c_\tau}_k)\}$, $g_t = \{(g^t_1 \mid \mu^{g_t}_1),\dots,(g^t_l \mid \mu^{g_t}_l)\}$, where $c^\tau_i \ge 0$ $\forall i \in J_n$; $g^t_j \ge 0$ $\forall j \in J_l$).

Let us prove, that

$$H(C) \le H(C+\Delta C).$$

It is obviously, $C \prec C+\Delta C$. Such property of the characteristic comparator is proved (see the statement 2.7 of the part 2 from [1]): if $a \prec b$, then $H(A) \le H(B)$. So $H(C)$, where C is – „partial” value of objective function, which define the subset D (the formula (6.10) from [1]) and have the property of the functional $\xi(D) = H(C)$, and according to the theorem 1 is an estimation of the subset. As has to be proved.

The remark. Estimating of feasible sets, considered during solving of CTPP, as it is seen, does not use the particular formulation of this problem and so can be used for general problem (1), where D is the set of a fuzzy permutations.

Such statement is true.

The theorem 3. Presented BBM, applied for the problem (1), yields its solution. F_0 (the value of the objective function) and x_{rec} (the point, in which the value of the objective function is achieved) act as this solution.

The proof. Only buds D_i in which the condition $\nu_i(D_i) \prec F_0$ is not executed are excluded from the consideration according to the step 11.

If $F_{n_{rec}} \prec F_0$, then on the step 15 F_0 updates and becames equal to $F_{n_{rec}}$, it is achieved in the point x_{rec}.

So, by virtue of the estimation: $\nu(D) \prec F(x) \quad \forall x \in D$, of the pattern:

$$F_1 \succ \ldots \succ F_{n_{rec}}$$

and the rule of cutting off $\nu_i(D_i) \prec F_0$ we have: $F_{n_{rec}} \prec F(x) \quad \forall x \in D$. As has to be proved.

6. CONCLUSION

The algorithm of BBM for the problem of the minimization in fuzzy formulation are proposed and substantiated. The usage of BBM for CTPP with fuzzy data are illustrated

7. REFERENCES

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