Algebraic Geometry

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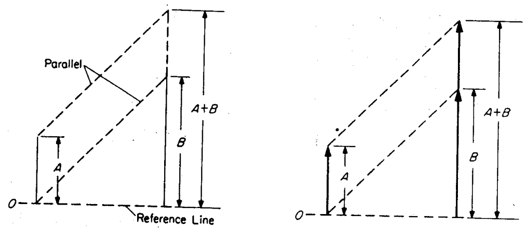
Abstract: *There are, algebraically speaking, three fundamental operations in arithmetic: 1) Addition and subtraction, 2) Multiplication and division, 3) finding powers and roots. Subtraction is actually negative addition; division is multiplication by the reciprocal of the divisor; and a root is a fractional power. If finding a power is considered as repeated multiplication, and if multiplication is considered as repeated addition, all operations may be reduced to a single basic one, which is addition. In turn, addition may be considered as merely a counting operation; this fact is the basis of electronic computation in modern digital machines. The graphic combination of the various arithmetic operations and their extension to graphic algebra will be the main topic. Each arithmetic operation can be represented graphically. The use of graphical arithmetic in itself is of relatively little importance. However when graphical arithmetic is applied to obtain an approximate solution to an algebraic equation, the topic can become very useful. Before the applications are considered, each of the arithmetic operations will be taken up separately*

Keywords*:* reciprocal – arithmetic series – phasor – normalize – modulus

1. GRAPHIC ADDITION AND SUBTRACTION

An arithmetic series may be written in the form $a_0 + a_1 + a_2 + ... + a_n$. The sum of the terms may be indicated by the notation $\sum a$. If the terms are considered to be collinear, then graphical addition could be performed as follows: A straight line of unlimited length (theoretically) is drawn, and then lengths representing the various terms are laid off successively either by making measurements directly with a scale or by using dividers. It is generally more practical, however, to consider the series as a number of parallel terms. Addition or subtraction is then based on the construction of a simple parallelogram. The process of adding two quantities by constructing a parallelogram is illustrated in Fig. 1(a), where two arithmetic quantities, A and B, have been added together in this fashion. The zero position is indicated by a horizontal reference line, and the positive direction is assumed to be upward. Therefore the negative direction would be downward. The position of a quantity along the horizontal reference line is unimportant. In Fig 1(b), the same addition has been performed, but the linear quantities are considered as parallel vectors. This procedure is obviously valid as long as the vectors have the same direction.

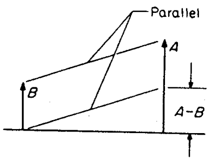


(a) Scalar Quantities **(b) Vector Quantities**

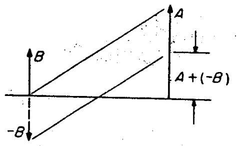
Fig. 1 – Graphical addition.

When a quantity is to be subtracted, the operation can be considered simply as the addition of an equal numerical quantity with the opposite sign. Thus, the subtraction A–B can be considered as the addition A+(–B), and the subtraction A–(–B) can be treated as the addition A+(+B) or simply A+B. Likewise, –A–B may be considered as –A+(–B), and –A– (–B) is the same as –A+(+B) or –A+B. If a positive quantity is to be subtracted from another positive quantity by a graphic method, the operation can be properly represented in at least two ways, which are consistent, with algebraic symbolism. One of these is illustrated in Fig. 2(a), where the horizontal line represents the zero position. First, the vectors representing both quantities are measured upward from the reference line. Next, a line is drawn so as to connect the terminal of the vector for the quantity B to be subtracted and the terminal of the vector for the other quantity A. Then, a line that is parallel to this connecting line is drawn from the beginning of the vector for B until it intersects the vector for A. The distance from the beginning of the vector for A to this intersection is the value A – B. Note that this value should be negative if the quantity B is numerically greater than A. In this case, the line from the beginning of the vector for B will intersect an extension of the vector for A below the reference line. Another way of subtracting one positive quantity from another graphically is indicated in Fig. 2 (b). In this method, the vector B is made negative by reversing its sense from the reference line, and then the vectors for A and B are added. Vector B has become vector –B by this reversal. Since A–B is the same as A+(–B), the result is valid. Either method is satisfactory, and the results should be the same.

The result of multiple additions, or of a combination of additions and subtractions, can be obtained quite easily by using vectors. The use of the vector notation in such a problem is mainly a matter of convenience, and whether or not this notation is used is immaterial. However, this method of combining quantities graphically will find many valuable applications in graphic manipulations, as well as in graphic thinking when accuracy need not be considered. Measurement could be made horizontally, or at some angle to the horizontal, as long as all vectorial distances are measured along parallel lines of action.



(a) As Subtraction of positive Quantities



(b) As addition of Negative Quantity
Fig. 2 – Graphical subtraction.

2. GRAPHIC MULTIPLICATION AND DIVISION

Since division may be considered a form of multiplication, these two operations are essentially the same. Geometrically, multiplication and division involve the use of similar triangles and therefore ratio and proportion. The fundamental idea underlying multiplication and division is shown geometrically in Fig 3. The corresponding sides of the similar triangles shown there form the proportion.

$$\frac{a}{d} = \frac{c}{b} \tag{1}$$

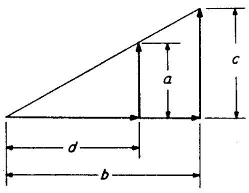


Fig. 3 – Graphic multiplication and division.

If the distance d is assumed to be unity, then equation (1) becomes

$$a = \frac{c}{b} \tag{2}$$

Either of these equations apparently indicates only division.

However the relationship in equation (1) can also be expressed as

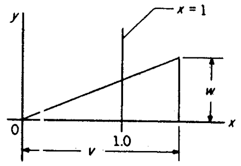
$$a * b = c * d \tag{3}$$

if d is unity,

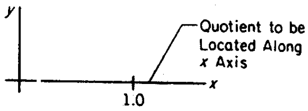
$$a * b = c \tag{4}$$

Hence, a product can also be indicated by considering the similar triangles. The distances a and b need not be measured at right angles, but the angle between the sides a and d must be equal to the angle between the sides b and c. Ordinarily, however, the use of right angles has an advantage. Likewise, it is advantageous to use horizontal and vertical directions, which are automatically at right angles. The use of vectors in the case of simple multiplication or division would be mainly a convenience. Geometric multiplication by the use of vectors must not be confused with vector multiplication in the sense that either a dot product, or a cross product is involved. Nor will geometric multiplication turn out to be a kind of phasor multiplication, it is simply an arithmetic operation performed graphically, and the result is a number which is actually nondirectional. For simple multiplication, the various factors are usually arranged as shown in Fig. (3).

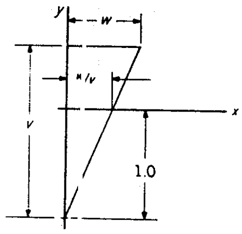
For division, the same arrangement serves equally well, as would be expected. The positioning of the various factors should not be rigidly established, however, because in a problem involving multiplication or division it is often advantageous to plan the arrangement so that the appropriate product or quotient will appear where it is wanted. It is a simple matter to select the position of the graphic manipulation so that the result will be in the desired position. For example, the quotient w/v in Fig. 4 (a) will appear along the line x = 1. However, let us suppose that this quotient is to be located along the x axis, as illustrated in Fig. 4(b). the rearrangement in Fig. 4(c) automatically puts the quotient in this desired location.



(a) Normal Arrangement



(b) Rearrangement



(c) Final result
Fig. 4 – Factor arrangement.

Another condition that must often be considered in obtaining a product or a quotient graphically is the fact that the scale in use may prevent the result form lying within the prescribed working space. Of course, this condition occurs more often in multiplication than in division. It is again a simple matter to select the scales in a problem so that the result will fall within a prescribed area. However, care must be exercised to make sure that the scale for measuring the result is proper in keeping with the scales used in measuring the given quantities. To consider a simple example, let it be required to multiply 4.1 by 63. These numbers are not compatible graphically, since one is about 15 times as large as the other. In order to make the two numbers graphically compatible, the modulus of the scale used for measuring 63 would have to be smaller than the other modulus. If a scale of 1 inch = 1 unit is used for measuring 4.1 and a scale of 1 inch = 10 units is used for 63, the multiplication would be performed as indicated in Fig. 5. The result shows that even with these scales it would be quite difficult to keep the product within a reasonable working space. Therefore, scales of 1 inch = 2.5 units and 1 inch = 20 units would be much more suitable.

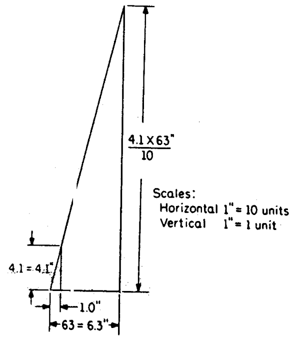


Fig. 5 – Scaled multiplication.

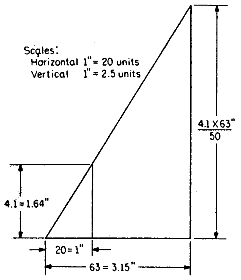


Fig. 6 – Fractional product scaling.

The result obtained by using these scales is shown in Fig. 6.

It is now necessary to determine what modulus should be used for reading the required product. The best way of doing this is to consider the equation representing the multiplication. If y denotes the length of the line representing the product, in inches, the equation showing what took place in Fig. 6 may be written as follows

$$y = \left(\frac{63}{20}\right)\left(\frac{4.1}{2.5}\right) \quad (5)$$

3. GRAPHIC METHODS FOR POWERS AND ROOTS

The discussion on graphic methods of finding powers and roots will be limited to powers for which the exponents are integers and to roots for which the indices are integers. Other powers and roots can be obtained graphically. However the manipulations become quite involved, and the number of applications does not warrant extending the process that far. Any power for which the exponent is an integer can be found by repetitive use of similar triangles, in the manner described for multiplication. This procedure is in keeping with the idea of a power being an extended multiplication. The method of finding the square of a quantity is illustrated in Fig. 7. The use of similar triangles in this case is in accordance with their previous use in multiplication, but their positions in the diagram are different. The use of a set of rectangular axes is a convenience and has application in many types of problems. If the axes are numbered in the fashion shown, the numbers will indicate the power to which the quantity has been raised by the graphic method.

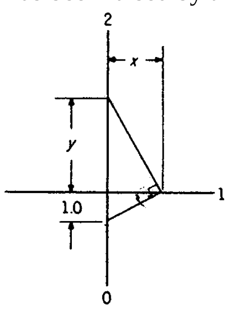


Fig. 7 – Graphically squared quantity.

The proportion arising from the similar triangles is:

$$\frac{y}{x} = \frac{x}{1} \quad (6)$$

From this equation,

$$y = x^2 \quad (7)$$

This relationship indicates that the construction is valid. If it is necessary to use smaller scales in order to keep the solution in the available space, it is a simple matter to change the number of units corresponding to 1 inch, as illustrated in Fig. 8. The other numbers must remain compatible, of course, but there should be no concern on this score because the only quantity involved is the original number to be raised to a power. In Fig. 8, a scale of 1 inch = 5 units has been chosen. If the quantity to be raised to the second power is designated as Q, then the proportion based on the similar triangles used would be

$$\frac{y}{Q} = \frac{Q}{5} \quad (8)$$

This equation leads to the relationship

$$5 * y = Q^2 \quad (9)$$

Therefore, the value measured in the y direction must be multiplied by 5 in order to obtain the correct result as shown in Fig. 8

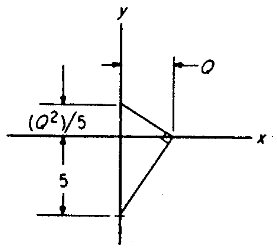


Fig. 8 – Scaling graphic power construction

The construction for obtaining the cube of a quantity would follow the pattern established for finding the square. It is simply necessary to extent the construction for the quantity to another quadrant, as shown in Fig. 9. The indicated similar triangles will produce the relationship.

$$\frac{w}{x^2} = \frac{x}{1} \quad (10)$$

Which shows that the construction is valid. It should be obvious that the process can be extended to find any desired power of a quantity merely by increasing the number of quadrants used around the origin. In each pair of similar triangles, the proper scaling must be taken into consideration. Again, moduli should be used in order to avoid an error in reading a result. One of the most advantageous types of scaling can be referred to as normalized scaling. In this kind of scaling, all values are kept equal to or less than unity through the use of proper scale factors or moduli. To achieve this result, the given quantity should always be expressed as the product of a number less than unity and some power of 10. The advtnage of this procedure, where results are to be measured, is that the diagram cannot go beyond the boundaries of a square each side of which is at a distance of unity from the origin. If it were required to raise a quantity to a power for which the exponent is a negative integer, the result could be obtained by use of a construction similar to that in Fig. 9. However, it would be necessary to proceed clockwise instead of counterclockwise. Also, in order that the scaling may be normalized, the quantity to be raised to the negative power must be expressed as the product of a number between 1 and 10 and some power of 10. The type diagram in Fig. 9 can therefore be used for any power for which the exponent is either a positive integer or a negative integer. If the quantity to be raised to the power is expressed in proper form, the construction will stay within the available space. The square root of a quantity, or any root for which the index is a power of 2, can be obtained by a construction based on a semicircle. Such a construction is shown in Fig. 10. However, this construction will not apply to a root for which the index is any other integer.

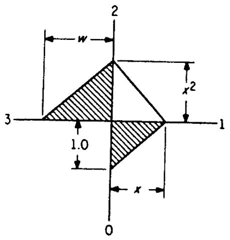


Fig. 9 – Finding cube graphically

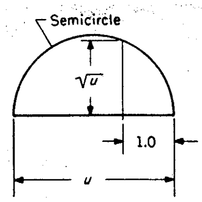


Fig. 10 – Square root construction

4. REAL ROOTS OF TWO – VARIABLE EQUATIONS

The general equation involving two variables is best expressed in the form

$$y = y(x) \qquad (11)$$

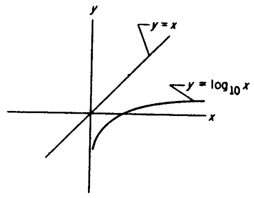
This equation simply states that y is some function of x. The real root or roots of such a function will be the value or values of x for which y is 0. If the function is plotted on the xy plane, each root will be the numerical value of x at a point where the curve crosses the x-axis. There are two general conditions where a graphic solution is quite valuable. One is where the functional relationship is known but a mathematical solution would be extremely complicated. The root or roots of such a function can be found quite easily by numerical methods if a computer is used. However, if great accuracy is not required, the graphical determination of the root or roots may be quite convenient. Of course, the graphic solution is also valuable as a check. The second use of a graphic solution is in the case where a series of points are plotted but an equation for the curve passing through the points cannot be readily determined. In such a case, a numerical method can be used, but its use may involve more work than is warranted by the required accuracy of the results and the use of a graphic method may possibly be more convenient. An example of the use of the first method will be shown. Let it be required to determine the roots of the equation

$$y - x = \log x \qquad (12)$$

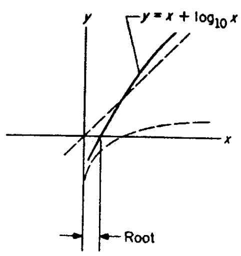
This equation can be plotted most easily if y is taken as two separate functions of x. It then becomes

$$y = x + \log x \qquad (13)$$

The right-hand member of Eq. (13) can be plotted as two functions of x which are superimposed on the same coordinate system. To clarify the procedure, it will be carried out in a piecemeal fashion. The equation contains a linear function of x and a logarithmic function of x. In Fig. 11(a), the linear and logarithmic functions have been plotted. In Fig. 11(b), the curve of their sum is shown. The intersection of this final curve and the x axis permits measurement of the root of the equation y = y(x) along that axis.



(a) Separated Functions



(b) Construction

Fig. 11 – Root of compound function

5. CONCLUSION

The use of graphical arithmetic in itself is of relatively little importance. However when graphical arithmetic is applied to obtain an approximate solution to an algebraic equation, the topic can become very useful.

6. REFERENCES

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