On the Stochastic Model of Clients Processing in Insurance Company with Time-Dependent Service Parameters

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Abstract: *A stochastic model of clients processing by the insurance company considered, model takes into account the limited duration of insurance contracts and the dependence of service rates on time. Closed exponential queuing network with one type of messages used as the model. The partial differential equation for the probability density function of the network state vector was obtained. A system of ordinary differential equations to predict the mean relative number of client requests at each stage of service was obtained. The examples of calculations were considered, their results are shown for illustration in graphical form, conclusions are made. A computer program that implements the calculation examples was implemented. The results of this paper could be applied in choosing the optimal strategy for the functioning of the insurance company and solution of control problems.*

Keywords**:** insurance company, insurance contract, claim, queuing network, asymptotic analysis, differential equation.

1. INTRODUCTION

The process of functioning of insurance company, concluding same type insurance contracts with its clients is considering [1]. It's supposed that maximum number of clients is K. For instance, it could be citizenship of a town in which company operates. m_4 of company employees engaged in contracting (insurers). Upon presentation of a claim, it goes through two stages of processing – assessment stage and payment stage. Assessment of claims involved m_1 employees (evaluators). Payment of the charges involved m_3 lawsuits cashiers. Each of the company's customers can be in one of the following states: C_2 – in waiting state, not going to submit an insurance claim; C_1 – in assessment claim state; C_3 – in the cash transactions state; C_4 – in the state of making of a contract. Let's also introduce state C_0, that means staying of company's potential customer in the "external environment". Assume that processing time of claims by evaluators is distributed exponentially with time-depending parameter $\mu_1(t)$, processing time of customers by cashiers is exponentially distributed with $\mu_3(t)$, processing time by insurers is exponentially distributed with $\mu_4(t)$.

Transition of some insurance claim from state C_0 to state C_4, as well as from C_2 to C_i, occurs at random instants of time independently on state of other claims, and regardless of the time so that probability of transition $C_0 \longrightarrow C_4$ on time interval $[t, t+\Delta t]$ equals $\mu_0(t)\Delta t + o(t)$, and probability of transition $C_2 \longrightarrow C_i$ on same time interval equals $\mu_{2i}(t)\Delta t + o(t)$. Here $\mu_0(t)$, $\mu_{2i}(t) = \mu_2(t) p_{2i}(t)$ – are transition rates, because of seasonality of insurance processes it's convenient to represent rates as periodic or piecewise constant functions of time; $p_{2i}(t)$ – time-dependent probabilities of transition from state C_2 to C_i, $0 \le p_{2i}(t) \le 1$, $i = \overline{0,1}$, $p_{20}(t) + p_{21}(t) = 1$.

Closed queueing network, fig. 1, with K messages circulating in it, which consists of five queueing systems S_0, S_1, S_2, S_3, S_4 could be used as a model of process of insurance claims processing described above, all queueing systems consist of K, m_1, K, m_3 and m_4 service lines accordingly [2]. Probabilities of messages (clients) transition between queueing systems are $p_{04} = p_{43} = p_{32} = p_{13} = 1$, $p_{2i}(t) \ne 0$, $i = \overline{0,1}$, in other cases $p_{ij} = 0$. Service disciplines of messages by queueing systems are FIFO.

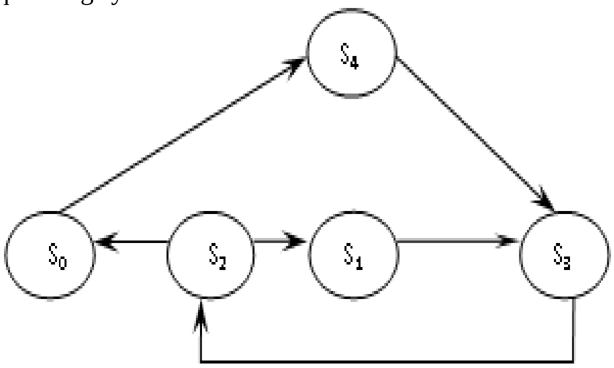

Fig. 1 – Network structure.

2. OBTAINING OF SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE MEAN RELATIVE NUMBER OF MESSAGES IN QUEUEING SYSTEMS

The state of the network at time instant t could be described by vector

$$k(t) = (k, t) = \big(k_1(t), k_2(t), k_3(t), k_4(t)\big), \qquad (1)$$

where $k_i(t)$ – total number of requests in state C_i, $i = \overline{1,4}$, then $k_0(t) = K - \sum_{i=1}^{4} k_i(t)$ – number of requests in state C_0. Obviously $k(t)$ is Markov process with continuous time and finite number of states.

Following transitions into state $k(t+\Delta t) = (k, t+\Delta t)$ of considered network during time Δt are possible:

– from state $(k - I_4, t)$ with probability

$$\mu_0(t)\big(k_0(t)+1\big)\Delta t + o(\Delta t) = \mu_0(t)\left(K - \sum_{i=1}^{4} k_i(t) + 1\right)\Delta t + o(\Delta t);$$

– from state $(k + I_4 - I_3, t)$ with probability

$$\mu_4(t)\min(m_4, k_4(t)+1)\Delta t + o(\Delta t);$$

– from state $(k + I_3 - I_2, t)$ with probability

$$\mu_3(t)\min(m_3, k_3(t)+1)\Delta t + o(\Delta t);$$

– from state $(k + I_2, t)$ with probability

$$\mu_2(t)p_{20}(t)(k_2(t)+1)\Delta t + o(\Delta t);$$

– from state $(k + I_2 - I_1, t)$ with probability

$$\mu_2(t)p_{21}(t)(k_2(t)+1)\Delta t + o(\Delta t);$$

– from state $(k + I_1 - I_3, t)$ with probability

$$\mu_1(t)\min(m_1, k_1(t)+1)\Delta t + o(\Delta t);$$

– from state (k, t) with probability

$$1 - \left[\mu_0(t)\left(K - \sum_{i=1}^{4} k_i(t)\right) + \sum_{\substack{i=1 \\ i\neq 2}}^{4} \mu_i(t)\min(m_i, k_i(t))\right]\Delta t + \\ + \mu_2(t)k_2(t)\Delta t + o(\Delta t);$$

– from all other states with probability $o(\Delta t)$.

Using law of total probability and passing to the limit $\Delta t \to 0$, one can obtain Kolmogorov system of difference-differentials equations for states probabilities

$$\begin{aligned}\frac{dP(k,t)}{dt} &= \mu_0(t)\left(K - \sum_{i=1}^{4} k_i(t)\right)[P(k - I_4, t) - P(k,t)] + \\ &+ \mu_0(t)P(k - I_4, t) + \\ &+ \mu_4(t)\min(m_4, k_4(t))[P(k + I_4 - I_3, t) - P(k,t)] + \\ &+ \mu_4(t)[\min(m_4, k_4(t)+1) - \min(m_4, k_4(t))]P(k + I_4 - I_3, t) + \\ &+ \mu_3(t)\min(m_3, k_3(t))[P(k + I_3 - I_2, t) - P(k,t)] + \\ &+ \mu_3(t)[\min(m_3, k_3(t)+1) - \min(m_3, k_3(t))]P(k + I_3 - I_2, t) + \\ &+ \mu_2(t)p_{20}(t)k_2(t)[P(k + I_2, t) - P(k,t)] + \\ &+ \mu_2(t)p_{20}(t)k_2(t)P(k + I_2, t) + \\ &+ \mu_2(t)p_{21}(t)k_2(t)[P(k + I_2 - I_1, t) - P(k,t)] + \\ &+ \mu_2(t)p_{21}(t)k_2(t)P(k + I_2 - I_1, t) + \\ &+ \mu_1(t)\min(m_1, k_1(t))[P(k + I_1 - I_3, t) - P(k,t)] + \\ &+ \mu_1(t)[\min(m_1, k_1(t)+1) - \min(m_1, k_1(t))]P(k + I_1 - I_3, t).\end{aligned} \quad (2)$$

Next let's consider case of large number of messages in the network, $K \gg 1$, and introduce vector of relative variables $\xi(t) = \left(\frac{k(t)}{K}\right)$, it's possible values belong to bounded closed set

$$G = \left\{x = (x_1, x_2, x_3, x_4) : x_i \geq 0,\ i = \overline{1,4}, \sum_{i=1}^{4} x_i \leq 1\right\},$$

where they are placed in nodes of 4-dimensional lattice at a distance $\varepsilon = \frac{1}{K}$ from each other. By increasing K "fill density" of set G by possible components of this vector is increasing as well, and it becomes possible to assume that is has continuous distribution with probability density $p(x,t) = K^4 P(xK, t)$, $x \in G$, where $p(x,t)$ is the meaning of the probability density of the random vector $\xi(t)$.

Let's denote by e_i 4-dimensional zero vector excepting of i-th component that equals ε, $i = \overline{1,4}$, $c(u) = \begin{cases} 1, u > 0, \\ 0, u \leq 0 \end{cases}$. Here $\min(u, v+1) = \min(u,v) + c(u-v)$, $c(u-v) = \frac{\partial \min(u,v)}{\partial v}$, because of $\min(u,v) = \begin{cases} v, u \geq v, \\ u, u < v \end{cases}$.

Rewriting system (2) for density $p(x,t)$, one get

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= K\mu_0(t)\left(1 - \sum_{i=1}^{4} x_i\right)[p(x - e_4, t) - p(x,t)] + \\ &+ \mu_0(t)p(x - e_4, t) + \\ &+ K\mu_4(t)\min(l_4, x_4)[p(x + e_4 - e_3, t) - p(x,t)] + \\ &+ \mu_4(t)\frac{\partial \min(l_4, x_4)}{\partial x_4} p(x + e_4 - e_3, t) + \\ &+ K\mu_3(t)\min(l_3, x_3)[p(x + e_3 - e_2, t) - p(x,t)] + \\ &+ \mu_3(t)\frac{\partial \min(l_3, x_3)}{\partial x_3} p(x + e_3 - e_2, t) + \\ &+ K\mu_2(t)p_{20}(t)x_2[p(x + e_2, t) - p(x,t)] + \\ &+ \mu_2(t)p_{20}(t)p(x + e_2, t) + \\ &+ K\mu_2(t)p_{21}(t)x_2[p(x + e_2 - e_1, t) - p(x,t)] + \\ &+ \mu_2(t)p_{21}(t)p(x + e_2 - e_1, t) + \\ &+ K\mu_1(t)\min(l_1, x_1)[p(x + e_1 - e_3, t) - p(x,t)] + \\ &+ \mu_1(t)\frac{\partial \min(l_1, x_1)}{\partial x_1} p(x + e_1 - e_3, t).\end{aligned}$$

Let's represent right-hand side of this system of equations up to terms of order of smallness ε^2. If $p(x,t)$ is twice differentiable by x, then the relations:

$$p(x \pm e_i, t) = p(x,t) \pm \varepsilon\frac{\partial p(x,t)}{\partial x_i} + \frac{\varepsilon^2}{2}\frac{\partial^2 p(x,t)}{\partial x_i^2} + o(\varepsilon^2),$$

$$p(x + e_i - e_j, t) = p(x,t) + \varepsilon\left(\frac{\partial p(x,t)}{\partial x_i} - \frac{\partial p(x,t)}{\partial x_j}\right) + \\ + \frac{\varepsilon^2}{2}\left(\frac{\partial^2 p(x,t)}{\partial x_i^2} - 2\frac{\partial^2 p(x,t)}{\partial x_i \partial x_j} + \frac{\partial^2 p(x,t)}{\partial x_j^2}\right) + o(\varepsilon^2),$$

$$i, j = \overline{1,4}.$$

Using it and also $\varepsilon K = 1$ one can obtain that probability density function $p(x,t)$ of network states vector satisfies the Fokker-Planck-Kolmogorov equation up to terms of order of smallness ε^2:

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=1}^{4}\frac{\partial}{\partial x_i}(A_i(x,t)p(x,t)) + \\ + \frac{\varepsilon}{2}\sum_{i,j=1}^{4}\frac{\partial^2}{\partial x_i \partial x_j}(B_{ij}(x,t)p(x,t)), \quad (3)$$

where

$$\begin{aligned}A_1(x,t) &= \mu_2(t)p_{21}(t)x_2 - \mu_1(t)\min(l_1, x_1); \\ A_2(x,t) &= \mu_3(t)\min(l_3, x_3) - \mu_2(t)x_2; \\ A_3(x,t) &= \mu_4(t)\min(l_4, x_4) + \mu_1(t)\min(l_1, x_1) - \\ &- \mu_3(t)\min(l_3, x_3); \\ A_4(x,t) &= \mu_0(t)(1 - \sum_{i=1}^{4} x_i) - \mu_4(t)\min(l_4, x_4);\end{aligned} \quad (4)$$

$$B_{11}(x,t) = \mu_2(t)p_{21}(t)x_2 + \mu_1(t)\min(l_1, x_1);$$

$$B_{22}(x,t) = \mu_3(t)\min(l_3, x_3) + \mu_2(t)x_2;$$
$$B_{33}(x,t) = \mu_3(t)\min(l_3, x_3) + \mu_4(t)\min(l_4, x_4);$$
$$B_{44}(x,t) = \mu_4(t)\min(l_4, x_4) + \mu_0(t)(1-\sum_{i=1}^{4} x_i);$$
$$B_{12}(x,t) = B_{21}(x,t) = -\mu_2(t)p_{21}(t)x_2;$$
$$B_{13}(x,t) = B_{31}(x,t) = -\mu_1(t)\min(l_1, x_1);$$
$$B_{23}(x,t) = B_{32}(x,t) = -\mu_3(t)\min(l_3, x_3);$$
$$B_{34}(x,t) = B_{43}(x,t) = -\mu_4(t)\min(l_4, x_4);$$
$$B_{14}(x,t) = B_{41}(x,t) = B_{24}(x,t) = B_{42}(x,t) = 0.$$

Equation (3) is Fokker-Planck-Kolmogorov equation for probability density function of Markov process $\xi(t)$. So components of vector of mean relative number of messages in queueing systems are $n(t) = (n_1(t), n_2(t), \ldots, n_{n+2}(t))$, where $n_i(t) = M\left(\frac{k_i(t)}{K}\right)$, $i = \overline{1,4}$. According to [3] these components satisfy following system of ordinary differential equations up to terms of order of smallness $O(\varepsilon^2)$:

$$n_i^{'}(t) = A_i(n(t)),\ i = \overline{1,4}, \qquad (5)$$

or using (4), we obtain the following system:

$$\begin{cases} n_1^{'}(t) = \mu_2(t)p_{21}(t)n_2(t) - \mu_1(t)\min(l_1, n_1(t)), \\ n_2^{'}(t) = \mu_3(t)\min(l_3, n_3(t)) - \mu_2(t)n_2(t), \\ n_3^{'}(t) = \mu_4(t)\min(l_4, n_4(t)) + \mu_1(t)\min(l_1, n_1(t)) - \\ \qquad - \mu_3(t)\min(l_3, n_3(t)), \\ n_4^{'}(t) = \mu_0(t)(1-\sum_{i=1}^{4} n_i(t)) - \mu_4(t)\min(l_4, n_4(t)). \end{cases} \qquad (6)$$

Right-hand sides of (6) are continuous piecewise linear functions. Such systems could be solved by splitting the phase space and finding solutions in the areas of linearity of the right-hand sides. For instance, in the area corresponding to the case of missed queues on client servicing stages $n_i(t) \le l_i$, $i = \overline{1,4}$, system (6) has the form

$$\begin{cases} \frac{dn_1(t)}{dt} = \mu_2(t)p_{21}(t)n_2(t) - \mu_1(t)n_1(t), \\ \frac{dn_2(t)}{dt} = \mu_3(t)n_3(t) - \mu_2(t)n_2(t), \\ \frac{dn_3(t)}{dt} = \mu_4(t)n_4(t) + \mu_1(t)n_1(t) - \mu_3(t)n_3(t), \\ \frac{dn_4(t)}{dt} = \mu_0(t)(1 - n_1(t) - n_2(t) - n_3(t) - n_4(t)) - \\ \qquad - \mu_4(t)n_4(t). \end{cases} \qquad (7)$$

For the numerical solution of differential equation of the form (7) mathematical software Maple could be applied.

3. EXAMPLE

Assume that $K = 20000$. All other parameters of insurance company functioning are as follows:

$$\mu_0(t) = 0.0005\sin(2\pi t/364) + 0.0009,$$
$$\mu_1(t) = 0.5\sin(2\pi t/364) + 2,$$
$$\mu_2(t) = 0.0004\sin(2\pi t/364) + 0.0009,$$
$$\mu_3(t) = 5\cos(2\pi t/364) + 12,$$
$$\mu_4(t) = \cos(2\pi t/364) + 2,$$
$$p_{21}(t) = 0.05\sin(2\pi t/364) + 0.09.$$

Program for Maple that can obtain numerical solution of the system of differential equations (7) is created. Fig. 2 shows the behavior of $n_3(t)$ graphically dependent on time. Remaining components of vector $n(t)$, that are solution of (7), could be investigated in similar way.

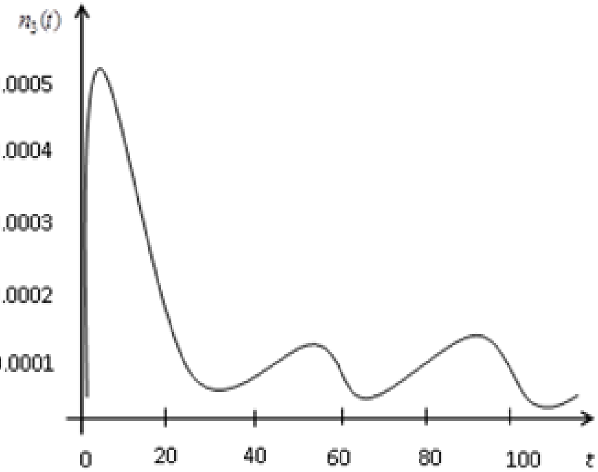

Fig. 2 – Plot of $n_3(t)$.

4. CONCLUSION

The above method of calculation of mean relative number of customer requests at service stages (messages in queueing systems) is valid only for high load of the network, i.e. for large values of K. The accuracy of the method increases with number of messages in queueing network. A computer program for calculation of examples is implemented. The results of this paper could be used in choosing the optimal strategy for the functioning of the insurance company, i.e. for solution of company management problems.

5. REFERENCES

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