ON STATIONARY DISTRIBUTION FOR MIGRATION PROCESSES OF A SPECIAL TYPE

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In the limited part of a plane the migration processes of a special type are considered. For the stationary probabilities an explicit formula of a vector-matrix form is obtained. It's suitable for calculation. The result is applied to the queues with repeated calls.

Keywords: migration process, stationary probabilities, queue with repeated calls.

We define the basic model under consideration in the paper as a two-dimensional Markov chain with continuous time $Q(t) = (Q_1(t); Q_2(t))$, $t \geq 0$ in bounded set of states $S = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, N\}$. Infinitesimal characteristics $q_{\alpha\beta}$, $\alpha, \beta \in S$, $\alpha \neq \beta$ of $Q(t)$ are given by a system of the relations:

under $\alpha = (i, j)$ and $i \neq m$

$$q_{\alpha\beta} = \begin{cases} \lambda_{ij}, & if \quad \beta = (i, j+1), \\ \mu_{ij}, & if \quad \beta = (i-1, j), \\ \nu_{ij}, & if \quad \beta = (i+1, j-1), \end{cases}$$

under $\alpha = (m, j)$

$$q_{\alpha\beta} = \begin{cases} \mu_{mj}, & if \quad \beta = (m-1, j), \\ \lambda_{mj}, & if \quad \beta = (m, j+1), \end{cases}$$

By agreement the rates of transitions which output from S are equal to zero

$$\mu_{0j} = \nu_{i0} = \nu_{mj} = \lambda_{mN} = 0.$$

The chains such as $Q(t)$ simulate Markov systems with repeated calls and the limited number of sources of repeated calls. So, under $\lambda_{ij} = \lambda_{mj} = \lambda$, $\mu_{ij} = i\mu$, $\nu_{ij} = j\nu$ we obtain the classical model of a system with repeated calls (see [1]) for which m is a number of servers, N is a maximal possible number of sources of repeated calls. By modification of ν_{ij} it is possible to obtain the model of a system with repeated calls and linear rate for the repeated attempts ([2]). The purpose of the presentation lies in the fact that for stationary probabilities π_{ij}, $(i, j) \in S$ of process $Q(t)$ we point out a representation which is convenient for their calculation and for the analytical analysis.

To formulate the basic result we introduce the notations:

for $j = 0, 1, \ldots, N-1$, $A(j) = \left\|a_{\alpha\beta}(j)\right\|_{\alpha,\beta=0}^{m-1}$ is tridiagonal matrix

$$a_{\alpha\beta}(j) = \begin{cases} \lambda_{\alpha j} + \mu_{\alpha j} + \nu_{\alpha j}, & \beta = \alpha, \\ -\lambda_{\alpha j}, & \beta = \alpha + 1, \quad \alpha = 0, 1, \ldots, m-2, \\ -\mu_{\alpha j}, & \beta = \alpha - 1, \quad \alpha = 1, 2, \ldots, m-1, \\ 0, & otherwise, \end{cases}$$

$$B(j) = \left\|b_{\alpha\beta}(j)\right\|_{\alpha,\beta=0}^{m-1},$$

$$b_{\alpha\beta}(j) = \begin{cases} \nu_{\alpha j+1}, & \beta = \alpha + 1, \quad \alpha = 0, 1, \ldots, m-2, \\ 0, & otherwise, \end{cases}$$

$$C(j) = \left\|c_{\alpha\beta}(j)\right\|_{\alpha,\beta=0}^{m-1},$$

$$c_{\alpha\beta}(j) = \begin{cases} \frac{\mu_{mj}}{\lambda_{mj}}\nu_{\alpha j+1}, & \beta = m-1, \quad \alpha = 0, 1, \ldots, m-1, \\ 0, & otherwise. \end{cases}$$

Via $A(N)$ we will designate a triangular matrix

$$A(N) = \begin{pmatrix} \mu_{1N} & 0 & 0 & \ldots & 0 & 0 \\ -(\nu_{1N} + \lambda_{1N}) & \mu_{2N} & 0 & \ldots & 0 & 0 \\ -\nu_{1N} & -(\nu_{2N} + \lambda_{2N}) & \mu_{3N} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\nu_{1N} & -\nu_{2N} & -\nu_{3N} & \ldots & -(\nu_{m-2N} + \lambda_{m-2N}) & \mu_{m-1N} \end{pmatrix}.$$

Also it will be necessary for us vectors:

$$\pi(j) = (\pi_{0j}, \pi_{1j}, \ldots, \pi_{m-1j}), \quad Q(j) = \frac{\pi(j)}{\pi_{0N}},$$

$$\nu(j) = (\nu_{0j}, \nu_{1j}, \ldots, \nu_{m-1j}), \quad j = 0, 1, \ldots, N,$$

$\bar{1}(m-1)$ is $(m-1)$-dimensional vector composed of 1, $e_i(m-1)$ is $(m-1)$-dimensional vector with i-th component is equal to 1 and the rest are equal to 0. Via $\bar{1}$, e_i we will designate the same vectors of dimensional m.

Theorem 1. *If for $Q(t)$ there exists stationary regime, $\mu_{iN} \neq 0$, $i = 1, 2, \ldots, m$, $\lambda_{mj} \neq 0$, $j = 0, 1, \ldots, N-1$, and $A(j)$, $j = 0, 1, \ldots, N-1$ has a reciprocal matrix then stationary probabilities π_{ij}, $(i, j) \in S$ has the following form*

$$\pi(N) = S_Q^{-1} Q(N), \quad \pi_{mN} = S_Q^{-1} \frac{1}{\mu_{mN}} Q(N)(\nu(N) + \lambda_{m-1N} e_m),$$

$$\pi(j) = S_Q^{-1} Q(N) T(N-1) \times \ldots \times T(j), \tag{1}$$

$$\pi_{mj} = S_Q^{-1} \frac{1}{\lambda_{mj}} Q(N) T(N-1) \times \ldots \times T(j+1) \nu(j+1),$$

$$j = 0, 1, \ldots, N-1,$$

where

$$S_Q = Q(N)\left\{[\bar{1} + \frac{1}{\mu_{mN}}(\nu(N) + \lambda_{m-1N}e_m) + \right.$$

$$\left. + \sum_{j=0}^{N-1} T(N-1) \times \ldots \times T(j+1)[T(j)\bar{1} + \frac{1}{\lambda_{mj}}\nu(j+1)]\right\},$$

$$Q(N) = \begin{pmatrix} 1 \\ A^{-1}(N)(\nu_{0N}\bar{1}(m-1) + \lambda_{0N}e_1(m-1)) \end{pmatrix},$$

$$T(j) = [B(j) + C(j)]A^{-1}(j).$$

For $M|M|m|N$-queue with repeated calls

$$B(j) = (j+1)\nu B, \quad C(j) = (j+1)\nu\frac{1}{\rho}C,$$

where $\rho = \frac{\lambda}{m\mu}$ is the system load due to primary calls,

$$B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ & & & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ & & \ldots & & \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Matrixes $A(j)$ may be represented in the form

$$A(j) = \Delta(j)[I - P(j)], \quad j = 0, 1, \ldots, N-1,$$

where $\Delta(j) = \|(\lambda + \alpha\mu + j\nu)\delta_{\alpha\beta}\|_{\alpha,\beta=1}^{m-1}$ is a diagonal matrix, $P(j) = \|p_{\alpha\beta}(j)\|_{\alpha,\beta=0}^{m-1}$,

$$p_{\alpha\beta}(j) = \begin{cases} \frac{\lambda}{\lambda+\alpha\mu+j\nu}, & \beta = \alpha+1, \quad \alpha = 0, 1, \ldots, m-2 \\ \frac{\alpha\mu}{\lambda+\alpha\mu+j\nu}, & \beta = \alpha-1, \quad \alpha = 1, 2, \ldots, m-1, \\ 0, & \textit{otherwise}. \end{cases}$$

Since the spectral radius of matrix $P(j)$, $j = 0, 1, \ldots, N-1$ is strictly less, than 1, then $A(j)$ has reciprocal matrix

$$A^{-1}(j) = [I - P(j)]^{-1}\Delta^{-1}(j) = [I + P(j) + P^2(j) + \ldots]\Delta^{-1}(j).$$

Thus, the conditions of Theorem 1 hold true and we obtain the following result.

Corollary 1. *Stationary probabilities for $M|M|m|N$-queue with repeated calls may be given in the form (1), where matrix*

$$T(j) = (j+1)\nu\left[B + \frac{1}{\rho}C\right][I - P(j)]^{-1}\Delta^{-1}(j), \qquad j = 0, 1, \ldots, N-1.$$

Evidently, the formula (1) contains an effective calculating scheme for stationary distribution. In the case one server ($m = 1$) the formula may be essentially simplified.

Corollary 2. *Stationary probabilities for $M|M|1|N$-queue with repeated calls have the form*

$$\pi(0,k) = S_Q^{-1} \prod_{j=0}^{k-1} \left(\rho \frac{j + \lambda/\nu}{j+1} \right), \tag{2}$$

$$\pi(1,k) = S_Q^{-1} \rho (1 + k \frac{\nu}{\lambda}) \prod_{j=0}^{k-1} \left(\rho \frac{j + \lambda/\nu}{j+1} \right), \quad k = 0, 1, \ldots, N, \tag{3}$$

where $S_Q = \sum_{m=0}^{N} [1 + \rho(1 + m\frac{\nu}{\mu})] \sum_{j=0}^{m-1} \left(\rho \frac{j+\lambda/\nu}{j+1} \right)$.

In closing we note that the simple formulas similar to (2), (3) can be obtained for other types of migration process $Q(t)$ defined by a specific dependence of its rates on a phase point of S.

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