A $BMAP|G|1$ QUEUE WITH RANDOMLY LIMITED FEEDBACK OPERATING IN A MARKOVIAN RANDOM ENVRIONMENT

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Feedback queues model, e.g., real-life systems, where the serving device is not absolutely reliable such that the repeated service of a customer can be required. In the present paper, the feedback queue with the batch markovian arrival and generally distributed service time is considered. Additionally, it is assumed that some markovian finite state random environment influences the arrival and the service processes as well as feedback probability. Moreover, it is assumed that the amount of service repetitions for a single customer is limited by a random variable. The stationary state distribution of the queue is calculated and numerical examples are presented.

Keywords: batch markovian arrival process, feedback, random environment.

1. INTRODUCTION

The interest to feedback queues goes from the fact that feedback queues describe real-life situations where the service of a customer should be repeated again because of non-satisfactory quality of a service or some other reason. Such situations take place in computer communication networks, production research and so on.

The first researcher who touched the problem of feedback queues was L.Takacs in [1]. Later, a lot of papers devoted to analogous queues were published. We can mention the names of J. Hunter, R. L. Disney, D. Koenig, V. Schmidt, M. A. Wortman, P. C. Kiessler, J. L. van den Berg, O. J. Boxma, J. A. C. Resing, M. C. Keane. In [2] the queue with the Batch Markovian Arrival Process (BMAP) is considered. However, in all these papers the similar feedback mechanism is considered. It is assumed there that an amount of repetitions of a service for a single customer is unlimited. In [3] another feedback mechanism is considered. It is assumed here that the amount of repetitions for a single customer is limited by a predefined constant. In the model presented the amount of repetitions of a service for a particular customer is limited by a random variable.

2. THE MATHEMATICAL MODEL

Random environment is identified by an irreducible continuous time Markov chain ν_t, $t \geq 0$. The state space of this process is given by $\{0, 1, \ldots, W\}$. The sojourn time of the process ν_t in the state ν has exponential distribution with the parameter λ_ν, $0 < \lambda_\nu < \infty$. When sojourn time expires, with probability $p^{(k)}_{\nu,\nu'}$ the process ν_t jumps into the state ν' and a batch of k customers arrives, $k \geq 0$, $\nu, \nu' = \overline{0, W}$. It is assumed that

$$p^{(0)}_{\nu,\nu} = 0, \sum_{k=0}^{\infty} \sum_{\nu'=0}^{W} p^{(k)}_{\nu,\nu'} = 1, \nu = \overline{0, W}.$$

It is suitable to keep information about the parameters of the process ν_t in the following matrices $D_k, k \geq 0$:

$$(D_k)_{\nu,\nu'} = \lambda_\nu p^{(k)}_{\nu,\nu'}, k \geq 1 \text{ and } k = 0, \nu \neq \nu', \qquad (D_0)_{\nu,\nu} = -\lambda_\nu, \nu = \overline{0, W}.$$

Denote the matrix generating function $D(z) = \sum_{k=0}^{\infty} D_k z^k, |z| \leq 1$.

The arrival process described above is the Batch Markovian Arrival Process (BMAP). Also the process ν_t is often referred as the underlying or directing process of the BMAP. Firstly, BMAP was investigated by Lucantoni [6]. For more details about the BMAP and related questions one can refer to the survey by S.Chakravarthy [7].

The service time distribution is given by $B^{(\nu)}(t), t \geq 0$ with expectation $b_1^{(\nu)} = \int_0^\infty t dB^{(\nu)}(t)$, where $\nu = \overline{0, W}$ is the state of the process ν_t at the service beginning epoch. Denote the following diagonal matrices:

$$B(t) = diag\{B^{(\nu)}(t), \nu = \overline{0, W}\}, t \geq 0, \; b_1 = diag\{b_1^{(\nu)}, \nu = \overline{0, W}\}.$$

After being served, a customer leaves the system forever with a fixed probability $1 - f_\mu$. With supplementary probability f_μ the customer immediately returns to the system and the service is repeated. Here $\mu = \overline{0, W}$ is the state of the process ν_t at the service completion epoch. Denote the diagonal matrix $F = diag\{f_\mu, \mu = \overline{0, W}\}$.

As it is told above, in earlier works on feedback models it was assumed that a number of service repetitions for a single customer was unlimited. Here we assume that service of a single customer cannot be repeated for more than $\Phi \geq 0$ times, where Φ is a random variable. Realization of this random Φ for a particular customer is defined at the epoch of the beginning of the first service of the customer. Its distribution is given by the probabilities $r_M = P\{\Phi = M\}$, $M \geq 0$, with the mean $\mathbf{E}\Phi = \sum_{M=0}^{\infty} M r_M$.

3. INVESTIGATION OF THE EMBEDDED MARKOV CHAIN

Consider two-dimensional continuous time process $\xi_t = \{i_t, \nu_t\}$, $t \geq 0$, where i_t is the number of customers in the system at the epoch t, ν_t is the state of the random environment at the epoch t. Consider discrete time embedded process $\xi_n = \xi_{t_n}$, $n \geq 1$, where t_n, $n \leq 1$, $0 \geq t_0 < t_1 < \ldots < t_n < \ldots$ are the epochs described as follows: (i) an arrival of the group of customers when the system is in idle state, (ii) departure of the customer from the system.

Analyzing the system operation, it easily can be seen that the embedded process ξ_n, $n \geq 1$ is a discrete time Markov chain.

Let us denote the stationary probabilities of the process ξ_n, $n \geq 1$ by:

$$\pi_{i,\nu} = \lim_{n \to \infty} P\{i_n = i, \nu_n = \nu\}, i \geq 0, \nu = \overline{0, W}.$$

Enumerating the states of the Markov chain ξ_n in lexicographic order, we introduce the following probability row vectors and their vector generating function :

$$\boldsymbol{\pi}_i = (\pi_{i,0}, \dots, \pi_{i,W}), i \geq 0,\ \boldsymbol{\Pi}(z) = \sum_{i=1}^{\infty} \boldsymbol{\pi}_i z^{i-1}, |z| \leq 1.$$

Lemma 1. *Equilibrium equations for the stationary probability vectors* $\boldsymbol{\pi}_i$, $i \geq 0$ *are as follows:*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0 V_0 + \sum_{j=1}^{i+1} \boldsymbol{\pi}_j Y_{i+1-j}, i \geq 0, \tag{1}$$

where

$$V_0 = 0,\ V_i = -D_0^{-1} D_i, i \geq 1,$$
$$Y_i = \sum_{M=0}^{\infty} r_M \left(\sum_{m=0}^{M-1} H_i^{(m)} (I - F) + H_i^{(M)} \right),$$
$$H_i^{(0)} = H_i,\ H_i^{(m)} = \sum_{j=0}^{i} H_j^{(m-1)} F H_{i-j}, m \geq 1, i \geq 0,$$

where H_i, $i \geq 0$ *are the coefficients of the following matrix expansion:*

$$H(z) = \sum_{i=0}^{\infty} H_i z^i = \int_0^{\infty} dB(t) e^{D(z)t}, |z| \leq 1.$$

Lemma 2. *Row-vector* $\boldsymbol{\pi}_0$ *and the vector generating function* $\boldsymbol{\Pi}(z)$ *satisfy the following equation:*

$$\boldsymbol{\Pi}(z)(Y(z) - zI) = \boldsymbol{\pi}_0 (I - V(z)), |z| \leq 1, \tag{2}$$

where

$$V(z) = \sum_{i=0}^{\infty} V_i z^i = -D_0^{-1} D(z) + I,$$
$$Y(z) = \sum_{i=0}^{\infty} Y_i z^i = \widehat{Y}(z)(H(z) - I) + I,$$
$$\widehat{Y}(z) = \sum_{M=0}^{\infty} r_M \sum_{m=0}^{M} \left(\widehat{H}(z)\right)^m,\ \widehat{H}(z) = H(z)F, |z| \leq 1.$$

Theorem 1. *The necessary and sufficient condition for the stationary distribution of the Markov chain* ξ_n, $n \geq 1$ *existence is follows:*

when $F = I$

$$\lambda < \frac{\boldsymbol{Y} \boldsymbol{b}_1 \boldsymbol{e}}{\mathbf{E}\Phi + 1}, \tag{3}$$

where

$$\boldsymbol{Y} H(1) = \boldsymbol{Y}, \boldsymbol{Y}\boldsymbol{e} = 1,$$

when $F \neq I$

$$\lambda < \boldsymbol{Z} \boldsymbol{b}_1 \boldsymbol{e}, \tag{4}$$

where

$$\boldsymbol{Z} H(1) = \boldsymbol{Z}, \boldsymbol{Z}(I - \widehat{H}(1)) \left(I - \sum_{M=0}^{\infty} r_M \left(\widehat{H}(1)\right)^{M+1} \right)^{-1} = 1.$$

Theorem 2. *Stationary probability vector* $\boldsymbol{\pi}_0$ *is the unique solution to the following system of linear equations:*

$$\begin{cases} \boldsymbol{\pi}_0\left(\sum\limits_{i=0}^{\infty} V_i G^i - I\right) = \mathbf{0}, \\ \boldsymbol{\pi}_0\left(I + V^{(1)}\left(Y^{(1)}\right)^{-1}\right)e = 1, \end{cases}$$

where $V^{(1)}$ *is the matrix first column of which is equal to* $-V'(1)\mathbf{e}$ *and other columns are equal to the corresponding columns of matrix* $(I - V(1))$,

$Y^{(1)}$ *is the matrix, first column of which is equal to* $(Y'(1)\boldsymbol{e} - \boldsymbol{e})$ *and other columns are equal to the corresponding columns of matrix* $(Y(1) - I)$,

G is the minimal nonnegative solution of the matrix equation $G = \sum_{i=0}^{\infty} Y_i G^i$,

$\mathbf{0}$ *is the row vector consisting of zeros,* $\boldsymbol{e}$ *is the column vector consisting of units.*

As the Markov chain ξ_n belongs to the class of $M/G/1$ type Markov chains then for finding the probability vectors $\boldsymbol{\pi}_i$, $i \geq 1$ we can use a recursion by V. Ramaswami [8]. Following V. Ramaswami, vectors $\boldsymbol{\pi}_i$, $i \geq 1$ are calculated as follows:

$$\boldsymbol{\pi}_i = \left(\boldsymbol{\pi}_0 \overline{V}_i + \sum_{j=1}^{i-1} \boldsymbol{\pi}_j \overline{Y}_{i+1-j}\right)\left(I - \overline{Y}_1\right)^{-1}, i \geq 1,$$

where

$$\overline{V}_i = \sum_{j=i}^{\infty} V_j G^{j-i}, \overline{Y}_i = \sum_{j=i}^{\infty} Y_j G^{j-i}, i \geq 0.$$

4. STATIONARY DISTRIBUTION AT ARBITRARY MOMENTS

Let us denote stationary probabilities of the process ξ_t, $t \geq 0$:

$$p_{i,\nu} = \lim_{t\to\infty} \boldsymbol{P}\{i_t = i, \nu_t = \nu\}, i \geq 0, \nu = \overline{0, W}.$$

Enumerating the states of the Markov chain ξ_t in lexicographic order, we introduce the following probability row vectors: $\boldsymbol{p}_i = (p_{i,0}, \ldots, p_{i,W})$, $i \geq 0$, and the vector generating function: $\boldsymbol{P}(z) = \sum\limits_{i=1}^{\infty} \boldsymbol{p}_i z^{i-1}$, $|z| \leq 1$.

Using the key theorem for Markovian renewal processes (see [9], [10]) in our case, it can be shown that the condition for the stationary state distribution existence of the process ξ_t, $t \geq 0$ coincides to the condition described in Theorem 1 for its embedded process ξ_n, $n \geq 1$.

Theorem 3. *The vector* $\boldsymbol{p}_0$ *and the vector generating function* $\boldsymbol{P}(z)$ *are calculated by:*

$$\boldsymbol{p}_0 = -\tau^{-1}\boldsymbol{\pi}_0 D_0^{-1}, \qquad \boldsymbol{P}(z)D(z) = \tau^{-1}\boldsymbol{\Pi}(z)(Y(z) - I),$$

where

$$\tau = -\boldsymbol{\pi}_0 D_0^{-1}\boldsymbol{e} + \boldsymbol{\Pi}(1)\widehat{Y}(1)b_1\boldsymbol{e}.$$

5. NUMERICAL EXAMPLES

To illustrate the work of the algorithms presented we consider the following numerical example. The arrival process is described by the following matrices:

$$D_0 = \begin{pmatrix} -0.517 & 0.1035 \\ 1.552 & -5.691 \end{pmatrix}, D_1 = \begin{pmatrix} 0.1035 & 0.129 \\ 1.811 & 0.259 \end{pmatrix}, D_2 = \begin{pmatrix} 0.0517 & 0.129 \\ 1.29 & 0.776 \end{pmatrix}.$$

The feedback probability f_0 is put equal to 0.7 and the probability f_1 is being changed during the experiment. The services times in both states are degenerately distributed with expectations $b_1^{(0)} = 0.2$, $b_1^{(1)} = 0.4$.

In the experiment four different feedback situations are considered:

(i) FB_1: $r_0 = 0.25$, $r_1 = 0.25$, $r_2 = 0.25$, $r_3 = 0.25$,

(ii) FB_2: $r_0 = 0$, $r_1 = 0.25$, $r_2 = 0.25$, $r_3 = 0.5$,

(ii) FB_3: $r_0 = 0$, $r_1 = 0$, $r_2 = 0.25$, $r_3 = 0.75$,

(ii) FB_4: $r_0 = 0$, $r_1 = 0$, $r_2 = 0$, $r_3 = 1$.

For all of these cases $r_M = 0$ for $M > 4$.

On the figure presented the dependency between the average queue length $L = \sum_{i=1}^{\infty} i\mathbf{p}_i\mathbf{e}$ and the feedback probability f_1 is examined.

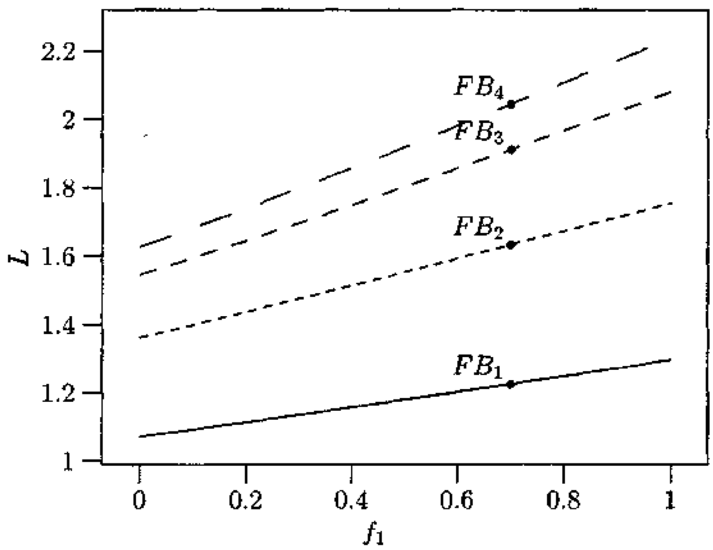

Fig. 1. Dependency between the average queue length L and the feedback probability f_1

6. CONCLUSION

The $BMAP|G|1$ queueing model with limited random feedback operating in random environment is investigated. The random environment has a finite state space. The state of the random environment affects the arrival process and the service time distribution. The stationary distribution of the multi-dimensional continuous time Markov chain describing the behavior of the system is calculated by means of an embedded Markov chain and the key theorem for Markovian renewal processes.

The results presented can be applied to the capacity planning of real-life objects and the performance evaluation in all situations where a repeated service can occur and the operation of the object is subject to some external influence.

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