G. Medvedev.
The Asset Pricing When the Interest Rates Are Differentiable Stochastic Processes.
Proceedings of the 11-th Annual International AFIR Symposium.
Toronto, 2001. Vol. 2, pp. 517 – 536.

The Asset Pricing When the Interest Rates Are Differentiable Stochastic Processes

Gennady Medvedev
Belarusian State University, Minsk,
Institute of Mathematics and Informatics PU, Czestochowa
E-mail: MedvedevGA@bsu.by

Abstract

This paper considers a problem of asset pricing for case when the short-term interest rate process does not have the markovian property. In this case the price can be determined also by state variables some of that are not observable. In the same time from the practical point of view, the mathematical expression for the asset price is acceptable for the participants of the market if it includes only observable variables. Therefore procedure of elimination from this mathematical expression of all not observable components of vector of state variables should be developed. In stochastic problems it is assumed to eliminate not observable indexes by taking the conditional expectation. Such approach is used in this paper. It is supposed that the interest rate process is differentiable but its mathematical derivative of some order is a diffusion process. In this case the values of this process at future times depend on values of process and its derivatives at present time. It means that there is a dependence on the process path. It is derived the expression for determination the asset price under these conditions. In this relation to usual formula for price some multiplier is added that depends on stochastic properties of mathematical derivatives of interest rate process. Extension of the Vasicek model on the differentiable processes is introduced. The comparison the bond price for this extension with bond price of standard Vasicek model is made.

The plan of paper is as follow. In Introduction the problem substitution is made and the state variables are determined. In Section 2 the no arbitrage condition for multi-factor model of term structure is given. The equation for the asset price at general multi-factor model is derived in Section 3. In next section it is shown as to eliminate not observable components of state variables. Section 5 contains the analysis of differentiable short-term interest rate processes. Special case of interest rate processes with one derivative is given in detail in Section 6. Equation for the asset price when the short-term interest rate process is differentiable is derived in Section 7. In Section 8 the extension of the Vasicek model is considered.

1. Introduction

In multi-factor models it is supposed that the world state, on which the asset price P depends, is determined by several quantities, which can be various quoted market indexes, and also connected with them not observable variables. Let's designate a vector composed of these quantities, through $R \in \mathrm{R}^M$, where $1 \le m \le M$, and M is complete number of market variables, on which the asset price depends, but m – number from these variables, which are quoted (are observed) in the market. In one-factor models of term structure of interest rates $M = m = 1$, i.e. the vector R is transformed in single variable and this single variable of state is the short-term (instant) interest rate.

From the practical point of view the mathematical expression for the asset price is acceptable for the participants of the market, if it includes only observable variables. Therefore procedure of exception from this mathematical expression of all not observable components of vector of state variable should be developed.

Something similar takes place and in one-factor model of term structure of the interest rates. The price of the discount bond with date of maturity T at the moment of time t is determined as its nominal value, discounted (in environment adjusted to risk) to the moment t, i.e. on an interval of time $[t, T]$. However the value of the short-term interest rate $r(t) = r$ is known only at the moment t, and its values at the future moments of time up to the moment T are unknown. The formula obtained in these conditions for the price of the bond and acceptable for the participants of the market is simply a conditional mathematical expectation (on probability measure adjusted by risk) of discounted nominal value on not observable future values of the short-term interest rates $\{r(s), t < s \le T\}$ under condition that $r(t) = r$. At use of the objective probability measure, generally speaking, the same occurs only nominal value is weighed in time by factor, that is dependent from the market price of risk (see Vasicek, 1977). Thus in this case actually the exception of values not observable variables from mathematical expression emanates by expectation of discounted nominal value of the bond on these not observable variables. This procedure can be accepted also in multi-factor model under determination of the price of financial asset. Only in this case not observable variables will be not only the future values of all M of state variables, but also the present value those $(M - m)$ state variables, which are not observed at the moment of time $t.$ Such procedure actually is a projection of function of the price given in “complete” space of state variables on subspace of the observable state variables. The operator of such projecting in a case of one-factor model is the calculation of conditional mathematical expectation. In this sense further we shall name as a *complete asset price* the mathematical expression for the asset price given in “complete” space of state variables, and as an *asset price* – its projection on subspace of observable state variables.

Let's assume, that the vector R changes over time satisfies to the following stochastic differential equation

$$dR = \mu(R, t)\, dt + \sigma\ (R, t)\, dW(t), \tag{1}$$

where $\mu(R, t)$ is a M-vector of drift of state variables, $\sigma(R, t)$ is $(M \times q)$-matrix them volatilities, and $dW(t)$ – vector of increments of the q-dimensional standard Wiener process with mutually independent components.

Let's assume further, that the complete asset price can be presented as the deterministic function from R, t and T, and this function is differentiable in respect to all variables necessary number of times. We hereinafter assume, that an asset does not pay any intermediate payments, and all payments are made in the maturity date T. Then the function of the complete asset price $P(R, t, T) \equiv P^{(T)}$ will be by the formula Ito to change during time according to the stochastic differential equation

$$dP^{(T)} = P^{(T)} \mu^{(T)}(t)\, dt + P^{(T)} \sigma^{(T)}(t)\, dW(t), \tag{2}$$

in which scalar factor of drift $\mu^{(T)}(t)$ and q-vector-row volatilities $\sigma^{(T)}(t)$ are brief designations of the following expressions (the superscript in brackets means that $\mu^{(T)}(t)$ and $\sigma^{(T)}(t)$ characterize an asset with the maturity date T, the superscript T without brackets will mean transposition):

$$\mu^{(T)}(t) = \frac{1}{P^{(T)}}\left(\frac{\partial P^{(T)}}{\partial t} + \frac{\partial P^{(T)}}{\partial R}\mu(R,t) + \frac{1}{2} tr\left(\frac{\partial^2 P^{(T)}}{\partial R^2}\sigma(R,t)\sigma^T(R,t)\right)\right),$$
$$\sigma^{(T)}(t) = \frac{1}{P^{(T)}}\frac{\partial P^{(T)}}{\partial R}\sigma(R,t). \tag{3}$$

For definiteness we shall notice that here $\partial P^{(T)} / \partial R$ is a M-vector-row and $\partial^2 P^{(T)} / \partial R^2$ is $(M \times M)$-matrix; tr A – trace of a matrix A.

2. The No Arbitrage Condition for Multi-factor Model of Term Structure

In the arbitrage theory of the asset pricing a usual way of deriving of the equation for determination of the price is the requirement of absence of arbitrage opportunities in the market, where n assets are traded that are differing only by the maturity dates T_j, $1 \le j \le n$. Actually the no arbitrage condition is also the partial differential equation for an complete asset price. Further for brevity instead of $P^{(T_j)}$, $\mu^{(T_j)}$, $\sigma^{(T_j)}$ we shall write accordingly $P^{(j)}$, $\mu^{(j)}$, $\sigma^{(j)}$, $1 \le j \le n$..

To obtain the no arbitrage condition in this market at first one composes a riskfree portfolio of assets and then requires that the yield of such portfolio in accuracy must be equal to the riskfree interest rate.

Let's assume that some investor has composed a portfolio of assets, investing at the moment of time t in an assets with the maturity date T_j the value of the size $V_j(t)$ (suppose that $V_j > 0$, if the investor purchases the assets, on which he in date T_j will receive the appropriate repayment, and $V_j < 0$, if the investor issues the obligations, i.e. in date T_j he will be obligated to pay the necessary cost). Then complete value of this portfolio at the moment of time t will be equal

$$v = \sum_{j=1}^{N} V_j(t) = \sum_{j=1}^{N} N_j(t) P(R,t,T_j) = \sum_{j=1}^{N} N_j(t) P^{(j)},$$

where N_j designates number of assets with maturity date T_j contained in a portfolio at the moment of time t. The increment of value of such portfolio for the time interval $(t, t + dt)$ will be equal

$$dv = \sum_{j=1}^{N} N_j(t) dP^{(j)} = \sum_{j=1}^{N} V_j(t) \frac{dP^{(j)}}{P^{(j)}}.$$

Using equation (2) that determines increment of asset price with maturity date T_j we receive stochastic differential equation for value of portfolio as

$$dv = \sum_{j=1}^{N} V_j(t) \frac{dP^{(j)}}{P^{(j)}} = \sum_{j=1}^{N} V_j(t) \mu^{(j)}(t) dt + \sum_{j=1}^{N} V_j(t) \sigma^{(j)}(t) dW(t). \tag{4}$$

Such portfolio will be riskfree, if the stochastic component of this equation will be equal to zero, i.e. $\sum_{j=1}^{N} V_j(t)\sigma^{(j)}(t) = 0$. For the further reasoning it is convenient to introduce *n*-vector-row $V(t) = (V_1(t), V_2(t), ..., V_N(t))$, $(N \times M)$-matrix $\partial P/\partial R$ with elements $(\partial P/\partial R)_{jk} = \partial \ln P^{(j)}/\partial R_k$, $1 \le j \le N$, $1 \le k \le M$, and $(N \times q)$-matrix $\sigma(t) = (\partial P/\partial R)\sigma(R, t)$ with rows $\sigma^{(j)}(t)$, $1 \le j \le N$, determined by the relation (3). Then a condition that the portfolio will be riskfree is the equality $V(t)\sigma(t) = 0$. It can be considered as the equation for determination of vector $V(t)$ of the investments in a riskfree portfolio. From here it is visible, that a necessary condition of existence of a riskfree portfolio is the inequality:

$$\text{rank}\, \sigma(t) = \min\{\text{rank}\,(\partial P/\partial R), \text{rank}\, \sigma(R, t)\} = \rho < N.$$

(We do not consider of course uninteresting version when the vector of the investments is zero.) So let necessary condition of existence of the riskfree portfolio $\rho < N$ takes place. Let's present a matrix $\sigma(t)$ in the block form (in a case, when $\rho < q$)

$$\sigma(t) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \tag{5}$$

where σ_{11} represents the nondegenerated block of the size $(\rho \times \rho)$. It in our conditions always can be reached by renumerating of assets. Other blocks are determined in appropriate way. According to this representation we shall write down in the block form a vector of the investments in the portfolio $V(t) = (V_1\ V_2)$. Then the condition of *existence of the riskfree portfolio* $V(t)\sigma(t) = 0$ takes a form

$$V_1\sigma_{11} + V_2\sigma_{21} = 0, \qquad V_1\sigma_{12} + V_2\sigma_{22} = 0. \tag{6}$$

From here it follows that between V_1 and V_2 there should be following linear dependence

$$V_1 = -V_2\sigma_{21}(\sigma_{11})^{-1}, \tag{7}$$

and the vector V_2 satisfies the equation

$$V_2(\sigma_{22} - \sigma_{21}(\sigma_{11})^{-1}\sigma_{12}) = 0, \tag{8}$$

which has a family of the not zero decisions as a matrix $(\sigma_{22} - \sigma_{21}(\sigma_{11})^{-1}\sigma_{12})$ is degenerated. Thus the variety of riskfree portfolios is determined by variety of the decisions V_2 of the equation (8), each of which uniquely determines appropriate V_1 by the formula (7), and also whole vector $V(t)$. In a case, when $\rho = q$, in a matrix (5) blocks σ_{12} and σ_{22} are absent, the second equation in (6) and equation (8) also are absent, and as V_2 can be chosen any not zero $(N-\rho)$-vector.

The no arbitrage condition requires, that yield of any riskfree portfolio in accuracy should be equal to the riskfree interest rate, which value at the moment of time t we shall designate by a symbol $r(t)$. As a rule, the riskfree interest rate is one of a component of a vector of state variables $R(t)$.

From relations (4) and (6) follows, that the equation of dynamics of value of a riskfree portfolio looks like

$$dv = \sum_{j=1}^{N} V_j(t)\mu^{(j)}(t)dt = (V_1\mu_1 + V_2\mu_2)\,dt = V_2(\mu_2 - \mu_1(\sigma_{11})^{-1}\sigma_{12})\,dt. \quad (9)$$

Here we again for brevity of records used the block structure of a vector $\mu(t)^{\mathrm{T}} = (\mu^{(1)}(t)\ \mu^{(1)}(t) \ldots \mu^{(N)}(t))^{\mathrm{T}} = (\mu_1^{\mathrm{T}}\ \mu_2^{\mathrm{T}})$. In order to there was no arbitrage should be satisfied condition

$$dv = \sum_{j=1}^{N} V_j(t)\mu^{(j)}(t)dt = v(t)\,r(t)\,dt = \sum_{j=1}^{N} V_j(t)r(t)dt. \quad (10)$$

If to enter the vector-columns r_1 and r_2 with identical components $r(t)$ in each of them with dimensions accordingly ρ and $(N-\rho)$ then last expression in (10) can be written down as

$$\sum_{j=1}^{N} V_j(t)r(t)dt = (V_1 r_1 + V_2 r_2)\,dt = V_2(r_2 - r_1(\sigma_{11})^{-1}\sigma_{12})\,dt.$$

Then from (9) and (10) the equality will follow

$$V_2[\mu_2 - r_2 - (\mu_1 - r_1)(\sigma_{11})^{-1}\sigma_{12}] = 0, \quad (11)$$

which should be carried out for any riskfree portfolio, i.e. for any not zero vector V_2 from variety that has been determined by the equation (8). This implies that every component of vector $\mu_2 - r_2 - (\mu_1 - r_1)(\sigma_{11})^{-1}\sigma_{12}$ should be equal to zero. This requirement is equivalent to a fact that for any j, that appropriates a component $V_j(t)$ of vector V_2, the equality takes place

$$\det\begin{pmatrix} \mu^{(j)}(t) - r(t) & \sigma_{12}^{(j)}(t) \\ \mu_1 - r_1 & \sigma_{11} \end{pmatrix} =$$

$$= (\mu^{(j)}(t) - r(t) - (\mu_1 - r_1)(\sigma_{11})^{-1}\sigma^{(j)}{}_{12}) \times \det\sigma_{11} = 0,$$

where $\sigma^{(j)}{}_{12}$ designates a row of a matrix σ_{12} with number j. From here follows, that the no arbitrage condition is carried out, if a matrix

$$\begin{pmatrix} \mu^{(1)}(t)-r(t) & \mu^{(2)}(t)-r(t) & \dots & \mu^{(N-1)}(t)-r(t) & \mu^{(N)}(t)-r(t) \\ \sigma_1^{(1)}(t) & \sigma_1^{(2)}(t) & \dots & \sigma_1^{(N-1)}(t) & \sigma_1^{(N)}(t) \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_\rho^{(1)}(t) & \sigma_\rho^{(2)}(t) & \dots & \sigma_\rho^{(N-1)}(t) & \sigma_\rho^{(N)}(t) \end{pmatrix} \quad (12)$$

has a rank ρ. In turn, it is equivalent to that the first line of a matrix (12) is a linear combination others ρ of lines. The elements of a matrix (12) are determined by equality (3), and $\sigma_k^{(j)}$ is k-th component of a row $\sigma^{(j)}(t)$. From here we obtain the no arbitrage condition in a final form

$$\mu^{(j)}(t) - r(t) = \sum_{k=1}^{\rho} \sigma_k^{(j)}(t)\lambda_k(R,t), \quad 1 \le j \le N. \quad (13)$$

Here $\lambda_k(R, t)$, generally speaking, can be any functions not dependent from j. It is necessary to notice, that the number of summands in the right part (13) is equal $\rho \le q$, i.e. not necessarily coincides with number q an independent stochastic component of the equation (1), as ρ is a rank of a matrix (5), composed from rows $\sigma^{(j)}(t)$, determined by equality (3). From the above analysis it does not follow of any recommendations concerning the form of functions $\lambda_k(R, t)$, therefore it is considered, that they should be given from any other reasons and are the same external factors, as functions of drift and volatility in the equation (1). It is accepted to name function $\lambda_k(R, t)$ as the *market price of risk* connected with influence of uncertainty, produced stochastic component with number k.

3. Equation for the Asset Price at General Multi-Factor Model

The equality (13) can be considered as the partial differential equation for the asset price with maturity date T_j if the explicit forms functions $\mu^{(T)}(t)$ and $\sigma^{(T)}(t)$ from equality (3) will be substituted in it. Let's enter for compactness of record ρ-vector-column $\lambda(R, t) = (\lambda_1(R, t)\ \lambda_2(R, t) \dots \lambda_\rho(R, t))^T$. Then the equality (13) can be written down as (for $\rho = q$)

$$\frac{\partial P^{(T)}}{\partial t} + \frac{\partial P^{(T)}}{\partial R}\mu(R,t) + \frac{1}{2} tr\left(\frac{\partial^2 P^{(T)}}{\partial R^2}\sigma(R,t)\sigma^T(R,t)\right) - r(t)P = \quad (14)$$
$$= \frac{\partial P^{(T)}}{\partial R}\sigma(R,t)\,\lambda(R, t).$$

The equation (14) is the equation for definition of the asset price with maturity date T in general statement for multi-factor model. To the equation (14) it is necessary to add a boundary condition $P(R, T, T) = \Psi(T)$, which determines payments in maturity date (execution of the contract) and reflects stipulated before a condition of the contract.

Unfortunately, the solution of the equation (14) in the explicit form for a general case can not be written down. It is possible to speak only about the solution in an analytical form for some special cases. Let's consider two of them.

1) The volatility matrix $\sigma(R, t)$ does not depend on R, vector functions $\mu(R, t)$ and $\sigma(R, t)\lambda(R, t)$ are linear in respect to R. In this case these functions are set by relations

$$\sigma(R, t)\sigma^T(R, t) = \delta, \quad \mu(R, t) = \beta + \alpha R, \quad \sigma(R, t)\,\lambda(R, t) = \eta + \xi R, \tag{15}$$

where α, β, δ, η, ξ are vectors and matrixes of the appropriate sizes, which in general case can depend on time t. We believe, that the riskfree interest rate $r(t)$ is determined by some combination of components of vector R or is one of these components. Therefore we shall enter also M-vector-row a such that $aR(t) = r(t)$. (If r(t) is one of a component of a vector R then the components a are zero with one exception: a component with number, which the riskfree interest rate $r(t)$ has in a vector R, is equal to unit.) Then the decision of the equation (14) is the function

$$P(R, t, T) = \Psi(T)\exp\{A(t, T) + B(t, T)R\}, \tag{16}$$

where scalar function $A(t, T)$ and M-vector-row $B(t, T)$ are found from the following differential equations (stroke designates derivative on time)

$$A' = B\,(\eta - \beta) - \tfrac{1}{2}\,B\,\delta B^{\mathrm{T}}, \tag{17}$$

$$B' = a + B\,(\xi - \alpha)\,, \tag{18}$$

with boundary conditions $A(T, T) = 0$ and $B(T, T) = 0$.

2) Matrix function $\sigma(R, t)\sigma^{\mathrm{T}}(R, t)$ and the vector functions $\mu(R, t)$ and $\sigma(R, t)\lambda(R, t)$ are linear functions in respect to R, i.e.

$$\sigma(R, t)\sigma^T(R, t) = \delta + R^{\mathrm{D}}\gamma^{\mathrm{D}}, \ \mu(R, t) = \beta + \alpha R, \quad \sigma(R, t)\,\lambda(R, t) = \eta + \xi R, \tag{19}$$

where α, β, δ, γ, η, ξ are vectors and matrixes of the appropriate sizes, which generally can depend on time t. The symbol D designates the transformation of a vector in a diagonal matrix, for example

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \dots \\ \gamma_M \end{pmatrix}, \quad \gamma^{\mathrm{D}} = \begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \gamma_M \end{pmatrix}. \tag{20}$$

In a considered case the function (16) also is the decision of the equation (14), however in this case scalar function $A(t, T)$ and M-vector $B(t, T)$ are found from the following differential equations

$$A' = B(\eta - \beta) - \tfrac{1}{2}\,B\delta B^{\mathrm{T}}, \tag{21}$$

$$B' = a + B(\xi - \alpha) - \tfrac{1}{2}\,B\gamma^{\mathrm{D}}B^{\mathrm{D}}, \tag{22}$$

with former boundary conditions $A(T, T) = 0$ and $B(T, T) = 0$.

These cases are widely known for the discount bonds paying a unit in date of maturity ($\Psi(T) = 1$) from papers, that were devoted the term structures for one-factor models with constant factors, when $M = q = 1$, $a = 1$, the functions A and B in (21) – (22) are scalar, and the parameters α, β, δ, γ, η, ξ transform to constants. The explicit form of functions $A(t, T)$ and $B(t, T)$, when $\gamma = 0$ и $\xi = 0$, was obtained in famous paper Vasicek (1977); in turn, the solution for the case $\delta = 0$ и $\eta = 0$ is found in widely known paper Cox, Ingerssol and Ross (1985); when all six parameters are non zero the decision is brought in Medvedev and Cox (1996). The detailed comparative analysis of functions $A(t, T)$ and $B(t, T)$, and also probability properties of processes of the short-term interest rate $r(t)$ determined by these three models, contains in Ilieva (2000).

The procedure of reception of the solution of the equation (14) in the considered cases is reduced that the equation Riccati (for vector B in a case (22)) at first should be solved. Unfortunately, in an analytical form it can be solved only in a scalar case for constant parameters. For more complex situations it is necessary to find the solution of the Riccati equation by numerical methods. The function A, if B is known, is determined by simple integration, that however can result in integrals which are not calculated explicitly. Below we shall consider one more case, when the decision of the equation (14) can be found in the closed form for a practically important case.

4. Elimination of not Observable Components of State Variables

The solution (16), that is found by described way, determines the "complete" asset price. It would be acceptable for the participants of the market, if all components of a vector R are observable. If some components of this vector are not observed in the market (earlier we determined them number as $(M - m)$ last component of a vector R), it is necessary to eliminate them from solution. For it we must still to find distribution of probabilities of these components and to calculate conditional mathematical expectation of the solution (16) on these components at the fixed observable variable condition. This will give the formula for the asset price acceptable for the participants of the market.

Let's consider this procedure for one important case of normal distribution of a vector of state variables $R(t)$. It means, that density of probabilities of a vector R has form

$$f(R, t) = \left((2\pi)^M \det \Sigma\right)^{-1/2} \exp\{-½\,(R - E)^{\mathrm{T}} \Sigma^{-1} (R - E)\}, \qquad (23)$$

where $E = E(t)$ is a vector of expectations of state variables, and $\Sigma = \Sigma(t)$ – matrix of their covariance. For convenience of the subsequent deriving we shall split the vector R on two parts: observable $G = (R_1\, R_2 \ldots R_m)^{\mathrm{T}}$ and not observable $H = (R_{m+1}\, R_{m+2} \ldots R_M)^{\mathrm{T}}$. According to this we shall present in the block form a vector E and matrix Σ:

$$E = \begin{pmatrix} E_g \\ E_h \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_g & \Sigma_{gh} \\ \Sigma_{hg} & \Sigma_h \end{pmatrix}, \qquad (24)$$

where E_g and E_h are mathematical expectations observable and not observable components respectively, Σ_g and Σ_h – according to their matrixes of covariance respectively, and Σ_{gh} and Σ_{hg} are matrixes mutual covariance of vectors observable and not observable components. Then density of probabilities of state variables can be written down in the form

$$f(G, H, t) = \frac{\exp\left\{-\frac{1}{2}\begin{pmatrix} G - E_g \\ H - E_h \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Sigma_g & \Sigma_{gh} \\ \Sigma_{hg} & \Sigma_h \end{pmatrix}^{-1} \begin{pmatrix} G - E_g \\ H - E_h \end{pmatrix}\right\}}{\sqrt{(2\pi)^M \det \begin{pmatrix} \Sigma_g & \Sigma_{gh} \\ \Sigma_{hg} & \Sigma_h \end{pmatrix}}}. \qquad (25)$$

This density of probabilities for our purposes is convenient to present in the form of product unconditional (for G) and conditional (for H at fixed G) densities

$$f(G)\,f(H\,|\,G) = \frac{\exp\left\{-\frac{1}{2}\left(G-E_g\right)^{\mathrm{T}}\Sigma_g^{-1}\left(G-E_g\right)\right\}}{\sqrt{(2\pi)^m \det\Sigma_g}} \times$$
$$\times \frac{\exp\left\{-\frac{1}{2}\left(H-\bar{E}_h\right)^{\mathrm{T}}\left(\Sigma_h-\Sigma_{hg}\Sigma_g^{-1}\Sigma_{gh}\right)^{-1}\left(H-\bar{E}_h\right)\right\}}{\sqrt{(2\pi)^M \det\left(\Sigma_h-\Sigma_{hg}\Sigma_g^{-1}\Sigma_{gh}\right)}}, \tag{26}$$

where for brevity the designation is used

$$\bar{E}_h = E_h - \Sigma_{hg}\Sigma_g^{-1}(G-E_g). \tag{27}$$

In considered case it is possible to write down the complete asset price (16) as

$$P(G, H, t, T) = \Psi(T)\exp\left\{A(t,T)+(B_g \quad B_h)\begin{pmatrix}G\\H\end{pmatrix}\right\}, \tag{28}$$

where a vector B is presented in the block form according to splitting of a vector of state variables into two parts G and H.

Now for deriving of the formula for asset price, which could be used in the market, it is necessary to calculate conditional expectation of function $P(G, H, t, T)$ on distribution of a vector H at the fixed vector G. Then we shall have

$$P(G, t, T) = \Psi(T)\exp\{A(t, T) + B_g G\}\, E_{G,t}\{\exp(B_h H)\} =$$
$$= \Psi(T)\exp\{A(t,T) + B_h(E_h + \Sigma_{hg}\Sigma_g^{-1}E_g) + ½\, B_h(\Sigma_h - \Sigma_{hg}\Sigma_g^{-1}\Sigma_{gh})B_h^{\mathrm{T}}\}\times$$
$$\times \exp\{(B_g + B_h\Sigma_{hg}\Sigma_g^{-1})G\}. \tag{29}$$

At last, if the observable and not observable market indexes are statistically independent among themselves (in our case it means, that $\Sigma_{gh} = 0$ and $\Sigma_{hg} = 0$), then we shall obtain

$$P(G, t, T) = \Psi(T)\exp\{A(t, T) + B_g G\}\times\exp\{B_h E_h + ½\, B_h \Sigma_h B_h^{\mathrm{T}}\}. \tag{30}$$

The not observable parameters in this formula determine last multiplier. By this multiplier the obtained formula differs from the formulae of the market asset price that are known from the literature,.

Let's remind, that the vector G is composed by observable market parameters, i.e. $G = (R_1\ R_2 \ldots R_m)^{\mathrm{T}}$. Thus in this formula for the price are used either the functions determined by accepted model (when it are supposed with normal distribution of state variables), or observable market parameters $R_1, R_2, \ldots R_m$. So at the given model of evolution of market indexes it can be used in real conditions.

5. Differentiable Processes of the Short-Term Interest Rates

In overwhelming number of cases the stochastic models of dynamics of the short-term interest rate are based on processes with independent increments (diffusion processes), which are continuous non differentiable Markov processes and are described by the equations of form (1), when the vector of a variable condition $R(t)$ degenerates in single variable $r(t)$. At the same time, the empirical evidences speak, that the real processes of the interest rates not always have Markov properties. This problem was discussed in many papers from different positions, that naturally required new ways of construction of models of dynamics of the short-term interest rate. We shall assume here the model offered in Medvedev (2000). The idea of this model is based on the assumption, that process of the short-term interest rate are differentiable $(M-1)$ times, and its $(M-1)$-th (mathematical) derivative is diffusion process satisfying to the appropriate stochastic differential equation.

In the considered case to obtain the equation for determination of the asset price admitting solution in the explicit form, we shall consider the linear stochastic differential equation of the order M in respect to r with volatility that is non dependent from r and continuous determined factors, i.e.

$$dr^{(M-1)}(t) - a_{M-1}(t)r^{(M-1)}(t)\,dt - \ldots - a_0(t)r(t)\,dt = b(t)dt + \sigma(t)\,dW(t), \quad (31)$$

so continuous mathematical derivatives $r^{(k)}(t)$, $0 \le k \le M-2$, have differentials $dr^{(k)}(t) = r^{(k+1)}(t)\,dt$, and mathematical derivative of the order $(M-1)$ has stochastic differential

$$dr^{(M-1)}(t) = a_{M-1}(t)r^{(M-1)}(t)\,dt + \ldots + a_0(t)r(t)\,dt + b(t)dt + \sigma(t)\,dW(t), \quad (32)$$

Let's notice, that at $\sigma(t) \equiv 0$ equations (32) become the homogeneous ordinary differential equation for the determined function, which has M derivative

$$\frac{d^M r}{dt^M} - \ldots - a_1(t)\frac{dr}{dt} - a_0(t)r(t) = 0\,. \quad (33)$$

It is possible to present a general solution of the equation (33) as

$$r(t) = \sum_{k=0}^{M-1} u_k(t,s)r^{(k)}(s), \qquad s \le t, \quad (34)$$

through values of process $r(t) \equiv r^{(0)}(t)$ and its mathematical derivatives $r^{(k)}(t)$, $1 \le k \le M-1$, at the initial moment of time s, and also some partial solutions $u_k(t, s)$, appropriate to a special set of the initial conditions: $r^{(k)}(s) = 1$, $r^{(j)}(s) = 0$, for all $j \ne k$. Let us assume now, that the values $\{r^{(k)}(s),\ 0 \le k \le M-1\}$ are the random variables. Then the function determined (34) will have the continuous (in square mean) mathematical derivative $r^{(k)}(t)$ up to the order $(M-1)$ inclusive and will be the unique solution of the homogeneous stochastic equation (31) with the random initial conditions $\{r^{(k)}(s),\ 0 \le k \le M-1\}$.

The decision of the stochastic differential equation (31) with the zero initial conditions is determined by the formula (see Øksendal, 1998)

$$r(t) = \int_s^t u(t,s)\sigma(s)dW(s), \quad s \le t, \tag{35}$$

where for any fixed s the function $u(t, s)$ of variable t, $t \ge s$, is the solution of the homogeneous differential equation (33) with the initial conditions

$$u(s, s) = 0,\ u^{(1)}(s, s) = 0,\ ...,\ u^{(M-2)}(s, s) = 0,\ u^{(M-1)}(s, s) = 1.$$

Thus, if to take the solution (35) of equations (31) with the zero initial conditions $\{r^{(k)}(s) = 0,\ 0 \le k \le M-1\}$ and to add to it the decision (34) homogeneous equations (33), then the obtained sum will give the solution of the equation (31) with the initial conditions $\{r^{(k)}(s),\ 0 \le k \le M-1\}$.

For use of this solution under deriving of the equation of determination of the asset price, that admits deriving of the formulas in an explicit form, it is more convenient to write the solution of the equation (33) in other form. Let's determine a M-vector of state variables R of the short-term interest rate $R(t)$ as follows

$$R_1(t) = r(t), \quad R_{k+1}(t) = \frac{d^k r(t)}{dt^k},\ 1 \le k \le M-1. \tag{36}$$

In these designations the infinitesimal increments of first $(M-1)$ components of a vector R will be determined as follows

$$dR_k(t) = R_{k+1}(t)\,dt, \quad 1 \le k \le M-1, \tag{37}$$

but as under the assumption the last component $R_M(t) = r^{(M-1)}(t)$ satisfies to the stochastic differential equation (33), formulae (33), (36) – (37) allow to write instead of the equation (31) the following system of M differential equations of the first order (in differentials)

$$\begin{aligned} &dR_1(t) = R_2(t)\,dt, \\ &\dots \\ &dR_{M-1}(t) = R_M(t)\,dt, \\ &dR_M(t) = a_{M-1}(t)R_{M-1}(t)\,dt + \dots + a_0(t)R_1(t)\,dt + b(t)dt + \sigma(t)\,dW(t). \end{aligned} \tag{38}$$

The solution of this system of the equations is convenient to present in the matrix form. For this purpose we shall rewrite (40) as

$$\begin{pmatrix} dR_1 \\ dR_2 \\ \dots \\ dR_M \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_0(t) & a_1(t) & a_2(t) & \dots & a_{M-1}(t) \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \dots \\ R_M \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ \dots \\ b(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ \dots \\ \sigma(t) \end{pmatrix} dW(t).$$

If now for compactness of record of bulky expressions to introduce matrix designations, this equation will be written down as

$$dR = \alpha(t)R\,dt + \beta(t)\,dt + \delta(t)\,dW(t), \tag{39}$$

where the designations are used

$$R(t) = \begin{pmatrix} R_1(t) \\ R_2(t) \\ \dots \\ R_{M-1}(t) \\ R_M(t) \end{pmatrix}, \quad \beta(t) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ b(t) \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \sigma(t) \end{pmatrix}, \tag{40}$$

$$\alpha(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_0(t) & a_1(t) & a_2(t) & \dots & a_{M-1}(t) \end{pmatrix}. \tag{41}$$

It means that equation (39) is appropriate to equation (1) in that $\mu(R, t) = \alpha(t)R + \beta(t)$, $\sigma(R, t) = \delta(t)$, however among M components only one has stochastic behavior. The solution of system (39) with the initial conditions

$$\{R_k(s) = r^{(k-1)}(s),\ 1 \le k \le M\} \tag{42}$$

in the integrated form is

$$R(t) = U(t, s)R(s) + \int_s^t U(t,\tau)\beta(\tau)d\tau + \int_s^t U(t,\tau)\gamma(\tau)dW(\tau), \tag{43}$$

where $U(t, s)$ is a fundamental matrix of the solutions of homogeneous system of the ordinary differential equations $R' = \alpha R$ (R' designates mathematical derivative of vector $R(t)$ in respect t).

Let's present some useful properties of a fundamental matrix of the solutions

$$\frac{\partial U(t,s)}{\partial t} = \alpha(t)\,U(t, s), \quad \frac{\partial U(t,s)}{\partial s} = -\,U(t, s)\,\alpha(s).$$

$U(t, s) = U(t, \tau)\,U(\tau, s)$ for any t, τ, s.

$U^{-1}(t, s) = U(s, t)$, $U(t, t) = I$, I – identity matrix.

$$\det U(t, s) = \exp \int_s^t \mathrm{tr}\ \alpha(\tau)d\tau$$

Besides if the matrix $\alpha(t) = \alpha$ is independent on t, then the matrix $U(t, s)$ depends only on one variable – difference $(t - s) = \tau$ but not on two variables, i.e. in this case $U(t,s) = U(t - s)$, $U(0) = I$, $U(t + s) = U(t)U(s)$, $U^{-1}(\tau) = U(-\tau)$, $\alpha U(\tau) = U(\tau)\alpha$.

Let's address now to the solution (43). As well as it was necessary to expect, because of linearity of the equation (31) its solution is the normally distributed stochastic process with conditional (at fixed $R(s)$) expectation and covariance matrix

$$E\{R(t)|R(s)\} = U(t, s)R(s) + \int_s^t U(t,\tau)\beta(\tau)d\tau, \tag{44}$$

$$\mathrm{Var}\{R(t)|R(s)\} = \int_s^t U(t,\tau)\delta(\tau)\delta^{\mathrm{T}}(\tau)U^{\mathrm{T}}(t,\tau)d\tau. \tag{45}$$

At practical applications one takes an interest usually in the processes for that so-called a steady regime exists, when on the enough large interval of time of process evolution its expectation and its covariance have no tendencies to unlimited increase. In this case integrals in equality (44) and (45) should exist at $s \to -\infty$. For this it is necessary that for every t a matrix $U(t, s) \to 0$ at $s \to -\infty$. Thus the dependence from $R(s)$ is lost and at limiting transition we have unconditional expectation and covariance matrix (if they exist)

$$E\{R(t)\} = \int_{-\infty}^t U(t,\tau)\beta(\tau)d\tau, \tag{46}$$

$$\mathrm{Var}\{R(t)\} = \int_{-\infty}^t U(t,\tau)\delta(\tau)\delta^{\mathrm{T}}(\tau)U^{\mathrm{T}}(t,\tau)d\tau. \tag{47}$$

Therefore, the problem of solving of the equation (39) turns to a task of finding of a fundamental matrix of the solutions $U(t, s)$. In general case it is impossible to find this matrix. However rather simple solutions are derived in that case, when the matrix (41) is independent from t, that is the coefficients in the equation (31) are constants. The following formula for $U(t, s)$ is fair in this case:

$$U(t, s) = U(t - s) = Ve^{A(t-s)}V^{-1}, \tag{48}$$

where

$$e^{A(t-s)} = \begin{pmatrix} e^{v_1\tau} & 0 & \dots & 0 \\ 0 & e^{v_2\tau} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{v_M\tau} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ v_1 & v_2 & \dots & v_M \\ \dots & \dots & \dots & \dots \\ v_1^{M-1} & v_2^{M-1} & \dots & v_M^{M-1} \end{pmatrix}. \tag{49}$$

Here $\{v_k, 1 \le k \le M\}$ are eigenvalues of a matrix α, i.e. roots of the equation $\det(\alpha - vI) = 0$. The formulas (48) – (49) are written down for the most simple case, when all eigenvalues are various. Let's notice, that for existence of the steady regime for process in this case, i.e. for existence of integrals in (46) – (47), it is necessary, that all eigenvalues of a matrix α either were negative, or had negative real parts (in case of complex numbers).

6. Example. Interest Rate Process Has One Derivative

Let process of the riskfree interest rate $r(t)$ has mathematical derivative $r'(t)$, which follows to diffusion process

$$dr'(t) = a_1 r'(t)\,dt + a_0 r(t)\,dt + b\,dt + \sigma\,dW(t), \tag{50}$$

that has a characteristic polynomial $v^2 - a_1 v - a_0 = 0$, which roots are

$$v_{1,2} = \frac{a_1}{2} \mp \sqrt{\frac{a_1^2}{4} + a_0}\ . \tag{51}$$

In the matrix form the equation (50) takes the form

$$\begin{pmatrix} dr(t) \\ dr'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix} \begin{pmatrix} r(t) \\ r'(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ b \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t). \tag{52}$$

From (51) it follows, that for existence of steady process of the riskfree interest rates it is necessary, that the factors a_0 and a_1 in the equation (50) were negative. The matrixes V and V^{-1} in (48) has form

$$V = \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \end{pmatrix}, \qquad V^{-1} = \frac{1}{v_2 - v_1} \begin{pmatrix} v_2 & -1 \\ -v_1 & 1 \end{pmatrix}, \tag{53}$$

and the fundamental matrix of the solutions is derived in the form

$$U(\tau) = \frac{1}{v_2 - v_1} \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{v_1 \tau} & 0 \\ 0 & e^{v_2 \tau} \end{pmatrix} \begin{pmatrix} v_2 & -1 \\ -v_1 & 1 \end{pmatrix} =$$

$$= \frac{1}{v_2 - v_1} \begin{pmatrix} v_2 e^{v_1 \tau} - v_1 e^{v_2 \tau} & -e^{v_1 \tau} + e^{v_2 \tau} \\ v_1 v_2 (e^{v_1 \tau} - e^{v_2 \tau}) & v_2 e^{v_2 \tau} - v_1 e^{v_1 \tau} \end{pmatrix}. \tag{54}$$

Solution of the equation (50) in form (43) is derived as

$$\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix} = \frac{1}{v_2 - v_1} \begin{pmatrix} v_2 e^{v_1 (t-s)} - v_1 e^{v_2 (t-s)} & -e^{v_1 (t-s)} + e^{v_2 (t-s)} \\ v_1 v_2 (e^{v_1 (t-s)} - e^{v_2 (t-s)}) & v_2 e^{v_2 (t-s)} - v_1 e^{v_1 (t-s)} \end{pmatrix} \begin{pmatrix} r(s) \\ r'(s) \end{pmatrix} +$$

$$+ \frac{b}{v_2 - v_1} \begin{pmatrix} (1 - e^{v_1 (t-s)})/v_1 - (1 - e^{v_2 (t-s)})/v_2 \\ e^{v_2 (t-s)} - e^{v_1 (t-s)} \end{pmatrix} + \tag{55}$$

$$+ \frac{\sigma}{v_2 - v_1} \int_s^t \begin{pmatrix} -e^{v_1 (t-u)} + e^{v_2 (t-u)} \\ v_2 e^{v_2 (t-u)} - v_1 e^{v_1 (t-u)} \end{pmatrix} dW(u)$$

under the initial conditions $r(s)$ and $r'(s)$ at the moment of time s.

As follows from (44) – (45), first two terms in (55) form conditional mathematical expectation $E\left\{\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix} \middle| \begin{pmatrix} r(s) \\ r'(s) \end{pmatrix}\right\}$. The covariance matrix $\mathrm{Var}\left\{\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix} \middle| \begin{pmatrix} r(s) \\ r'(s) \end{pmatrix}\right\}$ are independent on $\begin{pmatrix} r(s) \\ r'(s) \end{pmatrix}$ and are expressed in the form

$$\mathrm{Var}\left\{\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix} \middle| s\right\} = \begin{pmatrix} \mathrm{Var}\{r(t)|s\} & \mathrm{Cov}\{r(t), r'(t)|s\} \\ \mathrm{Cov}\{r(t), r'(t)|s\} & \mathrm{Var}\{r'(t)|s\} \end{pmatrix} =$$

$$= \left(\frac{\sigma}{v_2 - v_1}\right)^2 \int_0^{t-s} \begin{pmatrix} -e^{v_1\tau} + e^{v_2\tau} \\ v_2 e^{v_2\tau} - v_1 e^{v_1\tau} \end{pmatrix} \begin{pmatrix} -e^{v_1\tau} + e^{v_2\tau} & v_2 e^{v_2\tau} - v_1 e^{v_1\tau} \end{pmatrix} d\tau . \tag{56}$$

The explicit forms of elements of a matrix (56) are set by the following equalities

$$\mathrm{Var}\{r(t)|s\} = \left(\frac{\sigma}{v_2 - v_1}\right)^2 \left(\frac{4v_1v_2 - (v_1 + v_2)^2}{2v_1v_2(v_1 + v_2)} + \frac{e^{2v_1(t-s)}}{2v_1} + \frac{e^{2v_2(t-s)}}{2v_2} - \frac{2e^{(v_1+v_2)(t-s)}}{v_1 + v_2}\right),$$

$$\mathrm{Var}\{r'(t)|s\} =$$

$$= \left(\frac{\sigma}{v_2 - v_1}\right)^2 \left(\frac{4v_1v_2 - (v_1 + v_2)^2}{2(v_1 + v_2)} + \frac{v_1}{2} e^{2v_1(t-s)} + \frac{v_2}{2} e^{2v_2(t-s)} - \frac{2v_1v_2}{v_1 + v_2} e^{(v_1+v_2)(t-s)}\right),$$

$$\mathrm{Cov}\{r(t), r'(t)|s\} = \left(\frac{\sigma}{v_2 - v_1}\right)^2 \left(\frac{1}{2} e^{2v_1(t-s)} + \frac{1}{2} e^{2v_2(t-s)} - e^{(v_1+v_2)(t-s)}\right).$$

Let's notice, that at $t \to s$ all elements of a covariance matrix (55) tend to zero, that was necessary to expect, as in this case at fixed $r(s)$ and $r'(s)$ the uncertainty of a state disappears. The formulae for unconditional expectation and covariance matrix in case of negative roots v_1 and v_2 can be derived, if in expressions for these characteristics to pass to a limit at $s \to -\infty$. Thus if to take into account that the roots v_1 and v_2 are connected with coefficients of the equation (50) by relations $v_1v_2 = -a_0$, $v_1 + v_2 = a_1$, $(v_1 - v_2)^2 = a_1^2 + 4a_0$, and in this case on assumption $a_0 < 0$, $a_1 < 0$, then for unconditional expectation and covariance matrix of vector of state variables we shall receive the following expressions

$$E\left\{\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix}\right\} = \begin{pmatrix} b/|a_0| \\ 0 \end{pmatrix}, \quad \mathrm{Var}\left\{\begin{pmatrix} r(t) \\ r'(t) \end{pmatrix}\right\} = \frac{\sigma^2}{2|a_1|}\begin{pmatrix} 1/|a_0| & 0 \\ 0 & 1 \end{pmatrix}. \tag{57}$$

Thus, in the steady regime the interest rate $r(t)$ is independent from mathematical derivative $r'(t)$ (as them covariance are equal to zero) has expectation $b/|a_0|$ and variance $\sigma^2/|2a_0a_1|$, while mathematical derivative of the interest rates $r'(t)$ has zero expectation and variance $\sigma^2/|2a_1|$.

In terms of formula (29) $E_g = b/|a_0|$, $E_h = 0$, $\Sigma_g = \sigma^2/|2a_0a_1|$, $\Sigma_h = \sigma^2/|2a_1|$, $\Sigma_{gh} = 0$. If the observable market index is only the interest rate r and r' is not observable then in expression (29) $G = r$, $H = r'$.

7. Equation for the Asset Price when the Short-Term Interest Rate Process is Differentiable

In a case of the differentiable short-term interest rates their behavior over time has no Markov properties, therefore it is natural to consider, that the "complete" asset price depends not only on the interest rates, but also from its mathematical derivative, i.e. from all a component of a vector R. Formally this vector satisfies the equation (39), which is a special case of the equation (1). Therefore equation for definition of the asset price at the interest rate described by the equation (1), is valid for the interest rate described by the equation (39) too. Only remains to find out, how the features of the equation (39) effect on the form of the asset price, if this price can be find as explicit expression.

Let's address to consideration of the equation (14) for a case, when the process of change of the riskfree short-term interest rates is described by the equation (31) and state variables are short-term interest rate $r(t)$ and its mathematical derivatives $r^{(k)}(t)$, $1 \le k \le M-1$, which compose a vector R. From relations (39) - (41) follows, that in this case $q = 1$, and M-vector function of drift $\mu(R, t)$ and matrix of volatilities (it degenerates in a M-vector) $\sigma(R, t)$ in the equation (1) will have form

$$\mu(R,t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_0(t) & a_1(t) & a_2(t) & \dots & a_{M-1}(t) \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \dots \\ R_{M-1} \\ R_M \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ b(t) \end{pmatrix},$$

$$\sigma(R,t) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \sigma(t) \end{pmatrix}, \quad \sigma(R,t)\sigma^{\mathrm{T}}(R,t) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \sigma^2(t) \end{pmatrix}. \tag{58}$$

Some simplifications of the equation (14) follow from here. They are determined by a specific form (58) of vectors and matrixes

$$\frac{\partial P^{(T)}}{\partial R}\mu(R,t) = \sum_{k=1}^{M-1} R_{k+1}\frac{\partial P^{(T)}}{\partial R_k} + \left(\sum_{k=0}^{M-1} a_k(t)R_{k+1}\right)\frac{\partial P^{(T)}}{\partial R_M} + b(t)\frac{\partial P^{(T)}}{\partial R_M},$$

$$tr\left(\frac{\partial^2 P^{(T)}}{\partial R^2}\sigma(R,t)\sigma^T(R,t)\right) = \sigma^2(t)\,\frac{\partial^2 P^{(T)}}{\partial R_M^2}, \tag{59}$$

$$\frac{\partial P^{(T)}}{\partial R}\sigma(R,t)\,\lambda(R,t) = \sigma(t)\,\lambda(R,t)\,\frac{\partial P^{(T)}}{\partial R_M}.$$

In the right parts of equality (59) all variable and the functions are scalar, including the market price of risk $\lambda(R, t)$. Substituting expressions (59) in the equation (14), we receive it in the following form (we shall remind, that according to our designations here $r(t) = R_1$)

$$\frac{\partial P^{(T)}}{\partial t} + \sum_{k=1}^{M-1} R_{k+1}\frac{\partial P^{(T)}}{\partial R_k} + \left(\sum_{k=0}^{M-1} a_k(t)R_{k+1} + b(t) - \sigma(t)\lambda(R,t)\right)\frac{\partial P^{(T)}}{\partial R_M} +$$

$$+\ \frac{1}{2}\sigma^2(t)\,\frac{\partial^2 P^{(T)}}{\partial R_M^2} - R_1 P^{(T)} = 0. \tag{60}$$

To consider a problem of the solution of the equation (60) concretely it is necessary that the function $\lambda(R, t)$ was determined in the explicit form. As well as earlier in (15) we shall assume, that it is linear in respect to the vector R, i.e. $\sigma(R, t)\,\lambda(R, t) = \eta + \xi R$, $\xi = (\xi_1\ \xi_2 \ldots \xi_M)$. Thus as in a considered case all assumptions (15) are held, the expression (16) can be considered as the solution of the equation (60). Then the equations (20) and (21) for the functions $A(t, T)$ and $B(t, T) = (B_1\ B_2 \ldots B_M)$ take the following form

$$A' = (\eta - b)B_M - \tfrac{1}{2}\sigma^2 B_M^{\ 2}, \tag{61}$$

$$B_1' = 1 + (\xi_1 - a_0)B_M, \tag{62}$$

$$B_k' = (\xi_k - a_{k-1})B_M - B_{k-1}, \quad 2 \le k \le M. \tag{63}$$

Thus the complete function of the price of an active has an affine structure (in respect to R_k) and is determined by the formula

$$P(R, t, T) = \exp\ \{A(t,T) + rB_1\} \times \exp\left\{\sum_{k=2}^{M} B_k(t,T)R_k\right\}, \tag{64}$$

where variable R_k, $2 \le k \le M$, are not observed. Therefore it remains to compute the expectation (64) in respect to not observable variables. This results in to final formula of type (30). In case considered only first component of state variable vector is observed, the short-term interest rate $r(t)$, and other state variables are its mathematical derivatives.

Because the coefficients in the equations (61) – (63) are constants the functions $A(t, T)$ and $B(t, T)$ will depend on single argument, term to maturity $T - t = \tau$, i.e. $A(t, T) = A(\tau)$, and $B(t, T) = B(\tau)$. And also $\partial A/\partial t = -dA/d\tau$, $\partial B/\partial t = -dB/d\tau$. Further consider the following special case. Assume that market price of risk is a constant and is independent on short-term interest rate $r(t)$ and its mathematical derivatives. Then $\xi = 0$ and equations (61) – (63) together with their initial conditions compose system

$$\begin{aligned} &B_1'(\tau) = a_0 B_M(\tau) - 1, && B'(0) = 0, \\ &B_k'(\tau) = B_{k-1}(\tau) + a_{k-1} B_M(\tau), && B_k'(0) = 0, \quad 2 \le k \le M. \end{aligned} \tag{65}$$

The matrix of system (65) is complete coincide with matrix α determined by (41). Therefore it turns out that equations (65) and (39) are determined for the same matrix α. Then the fundamental matrixes of solutions are based on the same eigenvalues that for case in question are assumed different and negative (or have negative real parts for case of complex eigenvalues). So the functions $B_k(\tau)$ are well determined and have following properties: they are equal to zero at $\tau = 0$ and tend to some limit values at $\tau \to \infty$. The limit values $B_k(\infty)$ are easy found from (65) because of $B_k'(\infty) = 0$. So $B_M(\infty) = 1/a_0$, $B_k(\infty) = -a_k/a_0$, $1 \le k \le M-1$.

Now will continue analysis of example of previous section, where $M = 2$. The solutions of system (65) are

$$\begin{aligned} B_1(\tau) &= -\frac{a_1}{a_0} - \frac{1}{v_2 - v_1}\left(\frac{v_2}{v_1} e^{v_1 \tau} - \frac{v_1}{v_2} e^{v_2 \tau}\right), \\ B_2(\tau) &= \frac{1}{a_0} + \frac{1}{v_2 - v_1}\left(\frac{1}{v_1} e^{v_1 \tau} - \frac{1}{v_2} e^{v_2 \tau}\right), \end{aligned} \tag{66}$$

were v_1 and v_2 – roots of characteristic polynomial of system (65) that are determined by expression (51). The function A is computed as integral

$$A(\tau) = \int_0^\tau [(\eta - b)B_2(s) + (1/2)\sigma^2 B_2^2(s)]ds. \tag{67}$$

Substitution (66) and (67) in formula (57) with $B_g = B_1$ and $B_h = B_2$ will give the final formula for bond price in this example.

8. Extension of Vasicek Model

It is interesting to know how strongly the formula for asset price at case considered differs from appropriate formula in usual analysis. For it as example will consider the Vasicek model of short-term interest rates and its modification for approach considered. Vasicek (1977) supposed that the short-term interest rates follows the process with stochastic differential equation

$$dr = k(\theta - r(t))\,dt + \sigma dW(t), \tag{68}$$

were $k > 0$, $\theta > 0$, $\sigma > 0$. This process is observed and the discount bond price at the time t with $r(t) = r$ is determined by formula

$$P_V(r, \tau) = \exp\{A_V(\tau) + rB_V(\tau)\}, \tag{69}$$

where

$$A_V(\tau) = \int_0^\tau [(\lambda - k\theta)B_V(s) + (1/2)\sigma^2 B_V^2(s)]ds. \quad B_V(\tau) = (e^{-k\tau} - 1)/k. \tag{70}$$

Here λ is constant market price of risks. As it is known process (68) has values of the steady expectation $E[r(\infty)] = \theta$ and the steady variance $\text{Var}[r(\infty)] = \sigma^2/2k$.

Now we will construct a process (50) that would be equivalent to process (68) in such sense that it will have the same steady expectation $E[r(\infty)] = \theta$ and the same steady variance $\text{Var}[r(\infty)] = \sigma^2/2k$. For this it is sufficient that $a_0 = -k$, $a_1 = -1$, $b = k\theta$. This result in to equation

$$dr' = [k(\theta - r(t)) - r'(t)]\,dt + \sigma dW(t). \tag{71}$$

The equation (71) can be considered as extension of Vasicek model on the differentiable processes of short-term interest rates. For process (71) the roots of characteristic polynomial are

$$v_1 = -½(1 + \sqrt{1-4k}), \quad v_2 = -½(1 - \sqrt{1-4k}),$$

and functions B_1 and B_2 are computed by formulae

$$B_1(\tau) = -\frac{1}{k} - \frac{1}{\sqrt{1-4k}}\left(\frac{v_2}{v_1}e^{v_1\tau} - \frac{v_1}{v_2}e^{v_2\tau}\right),$$

$$B_2(\tau) = -\frac{1}{k} + \frac{1}{\sqrt{1-4k}}\left(\frac{1}{v_1}e^{v_1\tau} - \frac{1}{v_2}e^{v_2\tau}\right). \tag{72}$$

The complete analysis for any values of parameter k is out side of frameworks our analysis therefore we will suppose only that this parameter takes rather small values (empirical results confirm this, see for example Pearson and Sun (1994)), i.e. we suppose that $k < ¼$.

To obtain the analytical results for comparison with the Vasicek model we will consider the approximation of functions B_1 and B_2 that will be based on smallness of parameter k. Note that there is approximation $\sqrt{1-4k} = 1 - 2k + O(k^2)$. Therefore

$$v_1 = -1 + k + O(k^2), \quad v_2 = -k + O(k^2),$$

Then the functions B_1 and B_2 will be

$$B_1(\tau) = -\frac{1}{k} - ke^{(k-1)\tau} + \left(\frac{1}{k} + k\right)e^{-k\tau} + O(k^2),$$

$$B_2(\tau) = -\frac{1}{k} - (1+3k)e^{(k-1)\tau} + \left(\frac{1}{k} + 1 + 3k\right)e^{-k\tau} + O(k^2). \tag{73}$$

Comparison of these functions with (70) gives that

$$B_1(\tau) = B_V(\tau) + k\left(e^{-k\tau} - e^{(k-1)\tau}\right) + O(k^2) = B_V(\tau) + \varepsilon_1(\tau), \tag{74}$$

$$B_2(\tau) = B_V(\tau) + (1+3k)\left(e^{-k\tau} - e^{(k-1)\tau}\right) + O(k^2) = B_V(\tau) + \varepsilon_2(\tau),$$

where functions $\varepsilon_1(\tau)$ and $\varepsilon_2(\tau)$ can be considered as some adjustment functions that show difference functions B_1 and B_2 from function $B_V(\tau)$. These adjustment functions are nonnegative and such that $\varepsilon_{1,2}(0) = \varepsilon_{1,2}(\infty) = 0$. They are restricted from above by values

$$\max_\tau \varepsilon_1(\tau) < k + O(k^2), \quad \max_\tau \varepsilon_2(\tau) < 1 + k + O(k^2). \tag{75}$$

Thus the bond price in the Vasicek model is determined by expression

$$P_V(r, \tau) = \exp\{rB_V(\tau) + \int_0^\tau [(\lambda - k\theta)B_V(s) + ½\, \sigma^2 B_V(s)^2]ds\}, \tag{76}$$

where $B_V(\tau) < 0$ is determined in (70) while in modified model considered this price is determined by formula

$$P(r, \tau) = P_V(r, \tau) \times$$
$$\times \exp\left\{ r\varepsilon_1(\tau) + \frac{\sigma^2}{4k} B_2^2(\tau) + \int_0^\tau \left(k\theta - \lambda + \frac{\sigma^2}{2}(2B_V(s) + \varepsilon_2(s)) \right) \varepsilon_2(s) ds \right\}. \tag{77}$$

The last multiplier reflects the effect on bond price of extension of Vasicek model to the short-term interest rates processes that are not markovian processes and are differentiable one time. It should be waiting that first term in exponent will be not have an essential role and main contribution in difference from tradition formula will be give two last terms. Because they can have different signs the effect of modification should be careful investigation.

Reference

Cox, J., Ingersoll J., Ross S. (1985). A theory of the term structure of interest rates. *Econometrica.* Vol. 53, № 2. P. 385-407.

Ilieva, N. G. (2000). The comparative analysis of the term structure models of the affine yield class. *Proc. of the 10-th Intern. AFIR Symposium.* Tromsø. P. 367-393.

Medvedev G.A. (2000). The Processes with Dependent Increments as Mathematical Models of the Interest Rate Processes, *Proc. of the 10-th Intern. AFIR Symposium.* Tromsø. P. 483 – 505.

Medvedev, G. A., Cox S. H. (1996). The market price of risk for affine interest rate term structures. *Proc. of the 6-th Intern. AFIR Symposium.* Nuremberg. P. 913-924.

Øksendal, B. (1998). *Stochastic Differential Equations. An Introduction with Applications.* Berlin: Springer-Verlag.

Pearson, N. D. and T.-S. Sun. (1994). Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll and Ross Model. *Journal of Finance,* 49, 1279 – 1304.

Vasiček, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics.* Vol. 5. P. 177-188.