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Equations with δ-shaped coefficients: the finite-dimensional perturbations approach

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This paper is devoted to the study of the formal differential expression of the form

$$\Delta^2 u + a_0 \delta u + \sum_k a_k \frac{\partial \delta}{\partial x_k} + \sum_{1 \le k \le j} a_{kj} \frac{\partial^2 \delta}{\partial x_k \, \partial x_j}.$$

Approximations of the singular part by means a family of finite range operators are constructed and resolvent convergence of this family is investigated.

Keywords: multiplication of distributions; operators with δ-potential; self-adjoint extension; resolvent convergence; resonance

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1. Schrödinger operator with δ-potential

The stationary Schrödinger operator with singular potential, symbolically written as

$$-\Delta u + a\delta u, \tag{1}$$

where δ is the Dirac δ-function, a is constant, called the coupling constant, models scattering on a particle, located at the origin of coordinates. Expression (1) arises as formal limit as $\varepsilon \to 0$ of operators

$$L_\varepsilon u = -\Delta u + q_\varepsilon(x) u \tag{2}$$

in the case when the potential $q_\varepsilon(x)$ is supported at an $\varepsilon-$neighbourhood of zero, i.e. family (2) is an approximation of formal expression (1).

The mathematical difficulties that appear during the investigation of expression (1) are related to the fact that the product $\delta \cdot u$ in Equation (1) is not defined in the classical theory of distributions.

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Therefore, giving sense to expression (1), as a self-adjoint operator in the space $L^2(R^3)$ (which is usually necessary in quantum theory) requires overcoming some obstacles.

A mathematical interpretation of expression (1) was given by Berezin and Faddeev in [5], as follows: Let $\overset{\circ}{L}$ be the restriction of the Laplace operator $-\Delta$ onto the domain

$$D(\overset{\circ}{L}) = \{u \in H^2(\mathbb{R}^3), \quad u(0) = 0\},$$

where $H^2(\mathbb{R}^3)$ is the Sobolev space. Then $\overset{\circ}{L}$ is a symmetric, but non-self-adjoint operator on $L^2(R^3)$. The self-adjoint extensions $L^{(\alpha)}$ of the operator $\overset{\circ}{L}$ are considered as possible perturbations of the Laplace operator by potentials, supported at zero. These self-adjoint extensions $L^{(\alpha)}$ are naturally parameterized by a single real parameter $\alpha \in (-\infty, +\infty]$, and the value $\alpha = +\infty$ corresponds to the Laplace operator, i.e. $\alpha = +\infty$ if the perturbation does not influence the operator.

Expression (1) by itself does not contain the information, what self-adjoint extension $L^{(\alpha)}$ corresponds to the concrete situation. The concrete situation means here that an approximation L_ε of Equation (1) is given and the problem is to bring to light what self-adjoint extension corresponds to given L_ε. As a rule, in usual sense, the limit of L_ε does not exist and the resolvent convergence is considered here. Recall that $L_\varepsilon \to L^{(\alpha)}$ in a *resolvent sense*, if

$$\lim_{\varepsilon\to 0}(L_\varepsilon - \lambda I)^{-1} = (L^{(\alpha)} - \lambda I)^{-1}. \tag{3}$$

Friedman [9] studies approximations of the formal expression (1) by operators (2) with potentials

$$q_\varepsilon(x) = a(\varepsilon)\frac{1}{\varepsilon^3}V\left(\frac{x}{\varepsilon}\right), \tag{4}$$

where V is a given function. He demonstrates that if $V(x) \geq 0$ and $a(\varepsilon) > 0$, then the resolvents of perturbed operators L_ε converge to the resolvent of the non-perturbed Laplace operator $-\Delta$, i.e. perturbations of this sort, in the limit, do not influence the character of interaction. He also discovered the possibility of *resonance cases*, when the resolvents converge to the resolvent of a non-trivial extension $L^{(\alpha)}$. It takes place (in the concrete example considered) if and only if $a(\varepsilon) = b\varepsilon + c\varepsilon^2 + \cdots$ and the constant b belongs to some discrete set of negative values.

Similar results were obtained later in the case of the potentials (4) with arbitrary function V. About investigations of the expression (1), see [2].

The domain of $L^{(\alpha)}$, $\alpha \neq +\infty$, contains functions with a singularity at the point zero. Therefore, the description of expression (1) needs to give a sense to the product $\delta \cdot u$, where the function u is discontinuous at zero. To solve the problem of the multiplication of generalized functions in [4,6–8,11], new objects (new generalized functions) that form algebras were introduced, i.e. they allow well-defined multiplication and at the same time save general properties of generalized functions. It was found out that investigations of expression (1) can be interpreted as standard constructions in the theory of new generalized functions.

The construction of new generalized functions is based on some approximation of the distributions by families of smooth functions, indexed by a small parameter ε. In particular, different approximations of the δ-function have different properties with respect to multiplication, and in the theory of new generalized functions, these approximations are considered as different objects. It is essential in the problem under consideration – different approximations of the δ-function generate different self-adjoint extensions of $\overset{\circ}{L}$.

2. Rank-one perturbations of $-\Delta$

The most simple approximation of Equation (1) can be constructed with the help of a family of rank-one operators [1,3,10]. We remaind these results. By Φ we denote the set of functions φ from Schwartz space $S(\mathbb{R}^3)$, such that $\varphi(x) \in \mathbb{R}$ and

$$\int \varphi(x)\mathrm{d}x = 1.$$

Set $\varphi_\varepsilon(x) = (1/\varepsilon^3)\varphi(x/\varepsilon)$ and consider the family of operators

$$\delta_{\varphi\varepsilon}(u) = \varphi_\varepsilon(x) \int \varphi_\varepsilon(y)u(y)\mathrm{d}y. \tag{5}$$

If u is a smooth function, then $\delta_{\varphi\varepsilon}(u) \to u(0)\delta = \delta u$, i.e. family (5) gives an approximation of multiplication by the δ-function. Therefore, it is natural to consider the family

$$L_\varepsilon u = -\Delta u + a(\varepsilon)\delta_{\varphi\varepsilon}(u) \tag{6}$$

as an approximation of Equation (1).

For fixed $\varepsilon > 0$, the operator $\delta_{\varphi\varepsilon}$ is rank-one operator and the resolvent for L_ε can be constructed in explicit form. Investigations of the behaviour of these resolvents give the following result. Denote

$$M = \int\int \frac{\varphi(x-y)\varphi(x)}{4\pi\,\|y\|}\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

Theorem 1 *Resolvents $R(\lambda,\varepsilon) = (L_\varepsilon - \lambda I)^{-1}$, where L_ε have form* (6)*, tend to the resolvent of a nontrivial extension as $\varepsilon \to 0$, if and only if the coefficient $a(\varepsilon)$ admits an expansion*

$$a(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + o(\varepsilon^2) \tag{8}$$

and the resonance condition

$$a_1 M + 1 = 0 \tag{9}$$

is fulfilled. Under these conditions, the limit of resolvents is given by Equation (13)*.*

Sketch of proof. Denote $R(\lambda) = (-\Delta - \lambda I)^{-1}$. Let $\mu = (-\lambda)^{1/2}$, $\Re e\mu > 0$ and

$$E_\lambda(x) = \frac{1}{4\pi\,\|x\|}\mathrm{e}^{-\mu\|x\|}.$$

Here E_λ is a fundamental solution for $-\Delta - \lambda I$: $(-\Delta - \lambda I)E_\lambda = \delta$ and $R(\lambda)f = E_\lambda * f$.

For fixed ε

$$R(\lambda,\varepsilon)f = R(\lambda)f + \left[\frac{1}{a(\varepsilon)} + B(\varepsilon)\right]^{-1} G_\varepsilon(f)E_{\lambda\varepsilon},$$

where

$$E_{\lambda\varepsilon} = R(\lambda)\varphi_\varepsilon = E_\lambda * \varphi_\varepsilon, \quad B(\varepsilon) = \int E_{\lambda\varepsilon}\varphi_\varepsilon \mathrm{d}x, \quad G_\varepsilon(f) = \int [R(\lambda)f]\varphi_\varepsilon \mathrm{d}x.$$

Denote $u_0 = R(\lambda)f$. It is easy to check that $G_\varepsilon(f) \to u_0(0)$ and $E_{\lambda\varepsilon} \to E_\lambda$ as $\varepsilon \to 0$.

It follows from this that the limit of resolvents $R(\lambda, \varepsilon)$ exists if and only if there exists

$$\lim_{\varepsilon\to 0}\left[\frac{1}{a(\varepsilon)}+B(\varepsilon)\right]^{-1}=S(\lambda) \tag{10}$$

and in this case

$$R(\lambda, 0)=\lim_{\varepsilon\to 0}\ R(\lambda,\varepsilon)f=R(\lambda)f+S(\lambda)u_0(0)E_\lambda. \tag{11}$$

In particular, Equation (11) is the resolvent of a nontrivial extension (resonance case) if and only if $S(\lambda)\neq 0$.

The value $B(\varepsilon)$ admits the expansion

$$B(\varepsilon)=M\frac{1}{\varepsilon}-\frac{\mu}{4\pi}+\mathcal{O}(\varepsilon), \tag{12}$$

where M is given by Equation (7).

It follows that a non-zero limit in Equation (10) exists if and only if the coefficient $a(\varepsilon)$ has form (8), where $a_1=-1/M$. In this case,

$$R(\lambda,0)f=R(\lambda)f-\frac{4\pi}{4\pi\alpha+\mu}u_0(0)E_\lambda, \tag{13}$$

where $\alpha=a_2[M]^2$. Here $R(\lambda,0)$ is the resolvent of the self-adjoint extension $L^{(\alpha)}$.

3. Finite-dimensional perturbations of Δ^2

The approximations of Equation (1) including finite-rank operators were considered in [3] and new cases of the resonance were obtained. The finite-dimensional perturbation approach can be used for investigation of more complicated formal expressions. In this paper, we apply this approach to expression

$$\Delta^2 u+a_0\delta u+\sum_k a_k\frac{\partial\delta}{\partial x_k}u+\sum_{1\le k\le j}a_{kj}\frac{\partial^2\delta}{\partial x_k\,\partial x_j}u. \tag{14}$$

Let us introduce the multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $0\le\alpha_k\le 2$. We will use the standard notations $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $\partial_k=\partial/\partial x_k$, $D^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$. Denote $\Theta=\{\alpha : |\alpha|\le 2\}$. We remark that the set Θ contains 10 elements.

Then we can write Equation (14) in the form

$$\Delta^2 u+\sum_{\alpha\in\Theta}a_\alpha(D^\alpha\delta)u. \tag{15}$$

The family of admissible self-adjoint operators can be specified by Berezin–Faddeev's approach.

THEOREM 2 *Let $\overset{\circ}{L}$ be the restriction of the operator Δ^2 onto the domain*

$$D(\overset{\circ}{L})=\{u\in H^4(\mathbb{R}^3)\colon (D^\alpha u)(0)=0,\ \alpha\in\Theta\}. \tag{16}$$

The family of all self-adjoint extensions of $\overset{\circ}{L}$ can be parameterized by the family of all self-adjoint matrices S of the dimension 10×10.

This theorem follows from the fact that for $\Re e\lambda > 0$ there exist exactly 10 linearly independent solutions of the equation

$$(\overset{\circ}{L})^* u - \lambda u = 0$$

in the space $L_2(\mathbb{R}^3)$; these solutions are functions of the form $D^\alpha E_\lambda$, $\alpha \in \Theta$, where E_λ is the fundamental solution for $\Delta^2 - \lambda I$.

Now let us consider approximations of the formal expression (15). The most natural approximation is

$$L_\varepsilon u = \Delta^2 u + q_\varepsilon u,$$

where

$$q_\varepsilon(x) = \sum_{\alpha\in\Theta} a_\alpha(\varepsilon)(D^\alpha \varphi_\varepsilon)(x)), \quad \varphi \in \Phi,$$

but it is very difficult to investigate the behaviour of the resolvents for this approximation.

However more simple approximations exist. A smooth function u fulfills

$$\frac{\partial \delta}{\partial x_k} u = -\frac{\partial u}{\partial x_k}(0)\delta + u(0)\frac{\partial \delta}{\partial x_k}.$$

It follows from this that the family of operator

$$T_{k\varepsilon} u(x) = -\int \frac{\partial \varphi_\varepsilon}{\partial x_k}(y) u(y)\mathrm{d}y \varphi_\varepsilon(x) + \int \varphi_\varepsilon(y) u(y)\mathrm{d}y \frac{\partial \varphi_\varepsilon}{\partial x_k}(x), \tag{17}$$

where $\varphi \in \Phi$, is some approximation of multiplication by $\partial\delta/\partial x_k$.

In a similar way, an approximation of multiplication by $\partial^2\delta/\partial x_k\, \partial x_j$ can be constructed.

The approximations obtained include linear functionals of the form

$$G_\alpha(u) = \int (D^\alpha \varphi_\varepsilon)(y) u(y)\mathrm{d}y, \quad \alpha \in \Theta.$$

Therefore, it is natural to consider approximations of the singular part of formal expression (15) of the form

$$T_\varepsilon u = \sum_{\alpha\in\Theta} F_\alpha(u) D^\alpha \varphi_\varepsilon,$$

where the functional $F_\alpha(u)$ has the form

$$F_\alpha(u) = \sum_{\beta\in\Theta} A_{\alpha\beta}(\varepsilon) G_\beta(u)$$

and $A_{\alpha\beta}(\varepsilon)$ are given functions that depend on ε.

Then the family

$$L_\varepsilon = \Delta^2 + T_\varepsilon \tag{18}$$

is an approximation of formal expression (15). For fixed $\varepsilon > 0$, the operator T_ε is a finite-rank operator and we can construct the resolvent in explicit form.

Let us introduce some notations. A family of functions $\mathbf{f} = \{f_\alpha, \alpha \in \Theta\}$ will be called a vector function. Denote by $\langle \mathbf{f}, \mathbf{g} \rangle$ the inner product:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{\alpha \in \Theta} f_\alpha g_\alpha.$$

A family of function $\mathbf{A}(\varepsilon) = \{A_{\alpha\beta}(\varepsilon) \colon \alpha \in \Theta,\ \beta \in \Theta\}$ will be called a matrix function. As usual

$$\mathbf{Af} = \sum_{\beta \in \Theta} A_{\alpha\beta} f_\beta.$$

Denote $R(\lambda) = (\Delta^2 - \lambda I)^{-1}$.

THEOREM 3 *Let $\varepsilon > 0$, L_ε be family* (18) *and* det $\boldsymbol{A}(\varepsilon) \neq 0$. *The resolvent $R(\lambda, \varepsilon) = (L_\varepsilon - \lambda I)^{-1}$ can be written in the form*

$$R(\lambda, \varepsilon) f = R(\lambda) f + \langle [\boldsymbol{A}(\varepsilon)^{-1} + \boldsymbol{B}(\varepsilon, \lambda)]^{-1} \boldsymbol{F}(\varepsilon), \boldsymbol{E}(\varepsilon) \rangle, \tag{19}$$

where $\boldsymbol{E}(\varepsilon) = \{E_\alpha(\varepsilon)\}$ is the vector function with the components $E_\alpha(\varepsilon) = R(\lambda) D^\alpha \varphi_\varepsilon$, $\boldsymbol{B}(\varepsilon, \lambda)$ is the matrix with the entries

$$B_{\alpha\beta}(\varepsilon, \lambda) = \int [E_\alpha(\varepsilon)] D^\beta \varphi_\varepsilon \mathrm{d}x$$

and $\boldsymbol{F}(\varepsilon) = \{F_\alpha(\varepsilon)\}$ is vector-function with the components

$$F_\alpha(\varepsilon) = \int [D^\alpha \varphi_\varepsilon] R(\lambda) f \, \mathrm{d}x.$$

It is easy to check that

$$\lim_{\varepsilon \to 0} E_\alpha(\varepsilon) = D^\alpha E_\lambda,$$
$$\lim_{\varepsilon \to 0} F_\alpha(\varepsilon) = (D^\alpha u_0)(0), \quad \text{where } u_0 = R(\lambda) f.$$

Therefore, the limit of resolvents (19) exists if there exists the limit of matrices

$$\lim_{\varepsilon \to 0} [\mathbf{A}(\varepsilon)^{-1} + \mathbf{B}(\varepsilon, \lambda)]^{-1} = \mathbf{S}(\lambda) \tag{20}$$

and then

$$R(\lambda, 0) f = \lim_{\varepsilon \to 0} R(\lambda, \varepsilon) f = R(\lambda) f + \langle \mathbf{S}(\lambda) \mathbf{u}_0,\ \mathbf{E}_\lambda \rangle.$$

If $\mathbf{S}(\lambda) = 0$, then $R(\lambda, 0) f = R(\lambda) f$ and we obtain the trivial extension.

Therefore, the resonance case problem is reduced to the question: what are conditions on matrix $\mathbf{A}(\varepsilon)$ under which the limit (20) exists and $\mathbf{S}(\lambda) \neq 0$.

4. Resonance matrices

Similar questions arise if we consider finite-rank perturbations of other differential expressions, which give approximations of differential expressions with δ-shaped coefficients. These questions are particular cases of the following general problem.

Let $\mathbf{B}(\varepsilon)$ be a given matrix function of the dimension $m \times m$ and we will calculate the limits

$$\lim_{\varepsilon\to 0}[\mathbf{R}(\varepsilon) + \mathbf{B}(\varepsilon)]^{-1} = \mathbf{S} \tag{21}$$

for different matrix functions $\mathbf{R}(\varepsilon)$ of the dimension $m \times m$.

We will say that matrix $\mathbf{R}(\varepsilon)$ is a *resonance matrix*, if limit (21) exists and $\mathbf{S} \neq 0$. The problem is to describe the set of all resonance matrices for given $\mathbf{B}(\varepsilon)$.

In Section 2, corresponding values $R(\varepsilon) = a(\varepsilon)^{-1}$, $B(\varepsilon)$ were scalar functions ($m = 1$) and the answer was simple, and the form of all resonance functions $R(\varepsilon)$ was obtained in theorem 1. If $\mathbf{R}(\varepsilon)$ and $\mathbf{B}(\varepsilon)$ are matrix functions, the problem is more complicated.

Let the matrix $\mathbf{B}(\varepsilon)$ have an expansion of the form

$$\mathbf{B}(\varepsilon) = B_{-1}\frac{1}{\varepsilon} + B_0 + B_1\varepsilon + \cdots . \tag{22}$$

As in the proof of Theorem 1, if $\mathbf{R}(\varepsilon)$ admits the expansion

$$\mathbf{R}(\varepsilon) = R_{-1}\frac{1}{\varepsilon} + R_0 + \cdots , \tag{23}$$

and the conditions

$$R_{-1} + B_{-1} = 0, \quad \det[R_0 + B_0] \neq 0, \tag{24}$$

hold, limit (21) exists and $\mathbf{S} = [R_0 + B_0]^{-1}$. Conditions (24) specify a subclass of resonance matrix.

But there exist a lot of resonance matrices for which $R_{-1} + B_{-1} \neq 0$.

Denote $\mathbf{D}(\varepsilon) = \mathbf{R}(\varepsilon) + \mathbf{B}(\varepsilon)$, $d(\varepsilon) = \det D(\varepsilon)$ and let $\mathbf{D}(\varepsilon)^\sharp$ be the associated matrix for $\mathbf{D}(\varepsilon)$, i.e. the matrix of all its algebraic complements. We have

$$[\mathbf{D}(\varepsilon)]^{-1} = \frac{1}{d(\varepsilon)}\mathbf{D}(\varepsilon)^\sharp.$$

For the matrices of form (22) and (23), there exist expansions

$$d(\varepsilon) = d_{-m}\frac{1}{\varepsilon^m} + d_{-m+1}\frac{1}{\varepsilon^{m-1}} + \cdots , \tag{25}$$

$$\mathbf{D}(\varepsilon)^\sharp = D_{-m+1}\frac{1}{\varepsilon^{m-1}} + D_{-m+2}\frac{1}{\varepsilon^{m-2}} + \cdots , \tag{26}$$

where numbers d_k and matrices D_k (depending on $\mathbf{R}$ and $\mathbf{B}$) can be calculated. In particular,

$$d_{-m} = \det[R_{-1} + B_{-1}], \qquad D_{-m+1} = [R_{-1} + B_{-1}]^\sharp.$$

We will say that *the resonance conditions of the order* p are fulfilled, if

$$d_{-m} = d_{-m+1} = \cdots = d_{-m+p} = 0, \quad d_{-m+p+1} \neq 0, \tag{27}$$

$$D_{-m+1} = D_{-m+2} = \cdots = D_{-m+p} = 0, \quad D_{-m+p+1} \neq 0. \tag{28}$$

THEOREM 4 *Let matrices* $\boldsymbol{B}(\varepsilon)$ *and* $\boldsymbol{R}(\varepsilon)$ *admit expansions* (22) *and* (23)*. The matrix* $\boldsymbol{R}(\varepsilon)$ *is a resonance matrix if and only if for some* $p = 0, 1, 2, \ldots$ *the resonance conditions of the order* p *are fulfilled and in this case*

$$\lim_{\varepsilon\to 0}[D(\varepsilon, \lambda)]^{-1} = \frac{1}{d_{-m+p+1}}D_{-m+p+1} \neq 0.$$

Observe that for different p we have different expressions for the limit matrix. In particular, in the case $p = 0$ we have the basic resonance:

$$\lim_{\varepsilon\to 0}[\mathbf{D}(\varepsilon)]^{-1} = \frac{1}{d_{-m+1}}[R_{-1} + B_{-1}]^{\sharp} \neq 0.$$

Conditions (24) correspond to the resonance of the order $p = m - 1$.

It follows that in order to describe the matrices $\mathbf{A}(\varepsilon)$, which generate non-trivial self-adjoint extensions we need to do the following actions: to construct expansion (22) for matrix $\mathbf{B}(\varepsilon, \lambda)$, to calculate coefficients in expansions (25) and (26) and to insert expressions obtained into the resonance conditions (27) and (28).

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