HAMILTONIAN PROPERTIES OF LOCALLY CONNECTED GRAPHS WITH BOUNDED VERTEX DEGREE

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ABSTRACT. We consider the existence of hamiltonian cycles for locally connected graphs with a bounded vertex degree. For a graph G, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum vertex degrees, respectively. We explicitly describe all connected, locally connected graphs with $\Delta(G) \leqslant 4$. We show that every connected, locally connected graph with $\Delta(G) = 5$ and $\delta(G) \geqslant 3$ is fully cycle extendable which extends the results of P.B. Kikust (Latvian Math. Annual **16** (1975) 33–38) and G.R.T. Hendry (J. Graph Theory **13** (1989) 257–260) on fully cycle extendability of connected, locally connected graphs with the maximum vertex degree bounded by 5. Furthermore, we prove that problem HAMILTON CYCLE for locally connected graphs with $\Delta(G) \leqslant 7$ is NP-complete.
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1. Introduction

In this paper, we study hamiltonian properties of locally connected graphs with a bounded vertex degree.

Graph G is *hamiltonian* if G has a *Hamilton cycle*, i.e., a cycle containing all vertices of G. A cycle C in a graph G is *extendable* if there exists a cycle C' in G (called the *extension* of C) such that C' contains all vertices of C plus a single new vertex. If such a cycle C' exists, we say that C *can be extended to* C'. If every non-Hamilton cycle C in G is extendable, then G is said to be *cycle extendable*. We say that G is *fully cycle extendable* if G is cycle extendable and each of its vertices belongs to a triangle of G. Clearly, any fully cycle extendable graph is hamiltonian.

It is well-known that the problem of deciding whether a given graph is hamiltonian, is NP-complete, and it is natural to search for conditions under which Hamilton cycles exist in graphs of special classes. Study of hamiltonicity conditions for graphs with a prescribed local structure is one of rather important directions in graph theory and has many applications in VLSI design, molecular biology, telecommunication and network connectivity. The first results in this area were obtained in the 70s and 80s of the last century by Chartrand, Pippert, Gould and Polimeni [6, 7], Kikust [17, 18], Oberly and Sumner [23], Clark [9]. The basis of the approach developed in these papers is a special

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graph property that being mapped onto a local structure of a graph leads to the existence of a Hamilton cycle. The main results in this direction are reviewed in surveys [11, 13, 20].

Recent research on hamiltonicity of locally connected graphs (Bielak [3]; Lai, Shao and Zhan [19]; Li et al [22]; Zhan [28]) study special types of local connectivity (triangularly, quadranqularly connected, and N_2-locally connected graphs). We consider the existence of hamiltonian cycles for locally connected graphs with a bounded vertex degree.

For graph-theoretic terminology not defined here, the reader is referred to [4]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A path in G with the end vertices u and v is called a *(u, v)-path.* Graph G is *connected* if there exists a (u, v) -path for any vertices u and v of G; otherwise, G is *disconnected.* A graph is *k-connected* ($k \geqslant 2$) if there are k vertex-disjoint paths between every pair of its vertices.

For a vertex u of G, the *neighborhood* $N_G(u)$ of u is the set of all vertices adjacent to u. The *degree* of u is defined as $\deg_G u = |N_G(u)|$. The subscript G will be omitted when the context is clear. For a subset X of $V(G)$, the set of all vertices adjacent to some vertex in X is denoted by $N(X)$, i.e., $N(X) = \bigcup_{u \in X} N(u)$. A graph (not necessarily finite) is *locally finite* if $|N(u)|$ is finite for each vertex u, i.e., if all its vertex degrees are finite. Note that we consider locally finite but not necessary finite graphs only in Section 2. It is assumed that graph is finite when hamiltonian properties are considered.

The notation $u \sim v$ ($u \not\sim v$, respectively) means that vertices u and v are adjacent (nonadjacent, respectively). For disjoint sets of vertices U and V, the notation $U \sim V$ ($U \not\sim V$, respectively) means that every vertex of U is adjacent (nonadjacent, respectively) to every vertex of V. In the case that $U = \{u\}$, we also write $u \sim V$ and $u \not\sim V$ instead of $\{u\} \sim V$ and $\{u\} \not\sim V$, respectively.

The *minimum* and *maximum degrees* of vertices in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\delta(G) = \Delta(G) = r$, then G is said to be *regular* of degree r, or simply *r-regular.* A 3-regular graph is also called a *cubic* graph. In any graph, a vertex of degree zero is called *isolated* and a vertex adjacent to all other vertices is called *dominating.*

The following notation will be used further. As usual, P_n and C_n denote the path and the cycle, O_n and K_n denote the empty and the complete graphs on n vertices, respectively. Let W_n denote the wheel on $n+1$ vertices, and $K_{1,1,q}$ denote the complete tripartite graph with two parts of size 1 and one part of size q. In addition, by $K_{m,n}$ we denote the complete bipartite graph with parts of size m and n. Graph $K_{1,n}$ is called a *star.* Let $P_{1,\infty}$ denote the *one-way infinite path*, i.e., a graph with $V(P_{1,\infty}) = \{x_k \mid k \in \mathbb{N}\}$ and $E(P_{1,\infty}) = \{x_k x_{k+1} \mid k \in \mathbb{N}\}$. Similarly $P_{\infty,\infty}$, denote the *two-way infinite path* with $V(P_{\infty,\infty}) = \{x_k \mid k \in \mathbb{Z}\}$, $E(P_{\infty,\infty}) = \{x_k x_{k+1} \mid k \in \mathbb{Z}\}$. Here $\mathbb{N}$ and $\mathbb{Z}$ are the sets of natural numbers and integers, respectively. A subgraph obtained from graph G by deleting an edge e is denoted by $G - e$. Let $\overline{G}$ be the complement of G. We denote by G^2 the *square* of graph G, i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most 2 in G. For graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, we denote by $G_1 + G_2$ the *join* of these graphs, i.e., a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1),\, v \in V(G_2)\}$. The notation $G \cong H$ means that graph G is isomorphic to graph H.

For a subset of vertices $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G[X]$. We will use the notation $G(v)$ for the subgraph of G induced by the neighborhood $N(v)$ of vertex $v \in V(G)$, i.e., $G(v) = G[N(v)]$.

Given a graph G, define the following set of graphs $\boldsymbol{N}(G) = \{G(u) \mid u \in V(G)\}$. Notice set $\boldsymbol{N}(G)$ does not contain isomorphic copies, i.e., for example, $\boldsymbol{N}(C_5) = \{O_2\}$. Let S be a finite set of finite graphs. If $S = \boldsymbol{N}(G)$ for a graph G, then the set S is called *realizable* and the graph G is called a *realization* of S. In the case that S consists of a single graph H and $\{H\} = \boldsymbol{N}(G)$ for a graph G, we say that the graph H is *realizable* and G is a *realization* of H.

A locally finite graph G is called *locally connected* if the neighborhood $N(v)$ of each its vertex $v \in V(G)$ induces a connected subgraph $G(v)$, other than the trivial graph K_1 of a single vertex.

One of the first results regarding the structure of locally connected graphs with $\Delta(G) \leqslant 4$ is the following condition of hamiltonicity by Chartrand and Pippert [6].

Theorem 1 ([6]). *Let G be a connected, locally connected graph with $\Delta(G) \leqslant 4$. Then either G is hamiltonian or $G \cong K_{1,1,3}$.*

While studying hamiltonicity, many related properties have also been heavily explored. Theorem 1 has stimulated further research; see [2, 9, 10, 15, 18, 23, 25, 27]. In particular, Kikust [18] shows that each connected, locally connected 5-regular graph is hamiltonian, while the case that $\Delta(G) \leqslant 5$ and $\Delta(G) - \delta(G) \leqslant 1$ is handled in the following statement by Hendry [14].

Theorem 2 ([14]). *Let G be a connected, locally connected graph with $\Delta(G) \leqslant 5$ and $\Delta(G) - \delta(G) \leqslant 1$. Then G is fully cycle extendable.*

We generalize the elegant results of Hendry [14] and Kikust [18], and show that connected, locally connected graphs with the maximum degree of 5 and the minimum degree of at least 3 are fully cycle extendable. Note that the class of connected, locally connected graphs with the maximum vertex degree bounded by 5 is rather representative since this class has at least as many members as the class of triangle-free cubic graphs. Indeed, if, similarly to [5], each vertex of an arbitrary triangle-free cubic graph is replaced by a triangle and each edge is replaced by a diamond $K_4 - e$, we obtain a 5-regular locally connected graph.

This paper is organized as follows. In Section 2, we explicitly describe all connected, not necessarily finite, locally connected graphs with $\Delta(G) \leqslant 4$. In Section 3, we show that every connected, locally connected graph with $\Delta(G) = 5$ and $\delta(G) \geqslant 3$ is fully cycle extendable which generalizes the results of Kikust [18] and Hendry [14] on fully cycle extendability of connected, locally connected graphs with maximum vertex degree bounded by 5. Furthermore, in Section 4 we prove that problem Hamilton Cycle for locally connected graphs with $\Delta(G) \leqslant 7$ is NP-complete.

2. Locally connected graphs with $\boldsymbol{\Delta(G)} \leqslant 4$

As follows from Theorem 2, finite connected, locally connected graphs under additional conditions $\Delta(G) \leqslant 4$ and $\Delta(G) - \delta(G) \leqslant 1$ possess cyclic properties stronger than hamiltonicity. Such properties result from the following classification theorem that extends Theorem 1 (and Theorem 2 in case of $\Delta(G) \leqslant 4$) by describing explicitly all connected (not necessarily finite) locally connected graphs with the maximum degree of at most 4.

Theorem 3. *Let G be a connected, not necessarily finite, locally connected graph with $\Delta(G) \leqslant 4$. The following statements hold:*

(i) *If $\Delta(G) = \delta(G)$, then $G \in \{C_n^2 \mid n \geqslant 3\} \cup \{P_{\infty,\infty}^2\}$;*
(ii) *If $\Delta(G) - \delta(G) = 1$, then $G \in \{P_4^2, K_5 - e, W_4, O_2 + P_4\}$;*
(iii) *If $\Delta(G) - \delta(G) = 2$, then $G \in \{K_{1,1,3}, K_2 + \overline{P_3}, H_1, H_2, H_3, H_4\} \cup \{P_n^2 \mid n \geqslant 5\} \cup \{P_{1,\infty}^2\}$ (See Figure 1 for graphs H_1, H_2, H_3, H_4).*

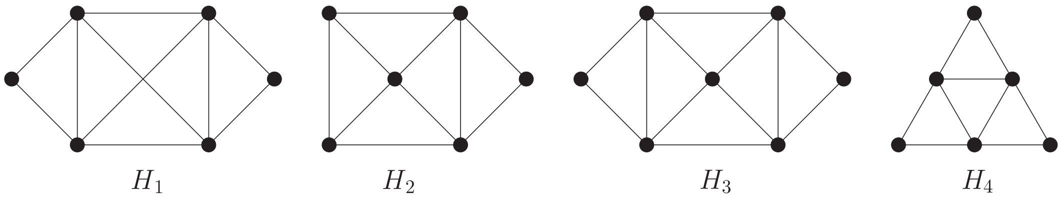

Fig. 1. Graphs H_1, H_2, H_3, H_4 from the formulation of Theorem 3

The proof of Theorem 3 is given in the end of this section, after several auxiliary statements.

Let $X \subseteq V(G)$ and $N(X)$ be the neighborhood of X in G. Denote by G_X the subgraph of G induced by set $N(X) \setminus X$. Note that set $V(G_X)$ may appear empty; in this case G_X is the null graph [26], i.e., G_X has no vertices and edges. A set $X \subseteq V(G)$ is called *maximal* in G, if there is no set $X' \subseteq V(G)$, such that $X \subset X'$ and $N(X') \cup X' = N(X) \cup X$.

The following statement holds.

Lemma 1. *For any maximal set $X \subseteq V(G)$ of a locally connected graph G, the subgraph G_X has no isolated vertices.*

Proof. Assume that there exists a maximal set $X \subseteq V(G)$ such that the subgraph G_X contains an isolated vertex u. Since $u \in N(X)$, it follows that there exists a vertex $v \in X$ adjacent to u. Since X is a maximal set, we deduce that $N(X) \cup X \neq N(X') \cup X'$, where $X' = X \cup \{u\}$. Thus, $u \sim w$ for some vertex $w \notin N(X) \cup X$, and since G is locally connected it follows that in the subgraph $G(u)$ there exists a (v, w)-path P such that $V(P) \cap (N(X) \setminus \{u\}) \neq \emptyset$. Let $z \in V(P) \cap (N(X) \setminus \{u\})$. Since $z \in N(u)$ there exists the edge uz in G (and also in G_X). This contradicts to the choice of vertex u as an isolated vertex of G_X. $\square$

We now turn to the regular, locally connected graphs. For a vertex $v \in V(G)$, let $D(v)$ denote the set of dominating vertices of the subgraph $G(v)$. The following statement holds.

Lemma 2. *For any vertex v of a regular, locally connected graph G, the subgraph $G(v) - D(v)$ contains no isolated vertices.*

Proof. For $v \in V(G)$, denote $X = \{v\} \cup D(v)$. Since graph G is regular, it follows that $N(X) \setminus X = N(v) \setminus D(v)$. Due to the same reason, any vertex that is not dominating in $G(v)$ is adjacent to a vertex not in $N(v) \cup \{v\}$. Therefore, set X is maximal, and due to Lemma 1 we derive that $G_X = G(v) - D(v)$ has no isolated vertices. □

The two following lemmas appear useful for the argument in this section.

Lemma 3. *Let G be a realization of the graph P_4. For a vertex $v \in V(G)$, denote the path P_4 induced by set $N(v)$ by (w_1, w_2, w_3, w_4). Then there exist a vertex $x \neq v$ such that $x \sim \{w_1, w_2\}$ and a vertex y, $y \notin \{v, x\}$ such that $y \sim \{w_3, w_4\}$.*

Proof. Since G is a 4-regular graph, there exists a vertex $x \in V(G) \setminus (N(v) \cup \{v\})$ such that $x \sim w_2$. The vertices x and w_1 are adjacent; otherwise the neighborhood $N(w_1)$ would induce a disconnected graph. Similarly, there exists a vertex $y \in V(G) \setminus (N(v) \cup \{v\})$, adjacent to w_3 and, therefore, to w_4. We show that the case $x = y$ is impossible. Assume the opposite; then the set $X = \{v, w_2, w_3, x\}$ is maximal. But graph G_X contains isolated vertices w_1 and w_4, a contradiction to Lemma 1. This proves the lemma. □

In a similar way, we can prove the following statement.

Lemma 4. *Let G be a realization of the set $\{K_2, P_3, P_4\}$. For any vertex $v \in V(G)$ of degree 4, denote the path P_4 induced by set $N(v)$ by (w_1, w_2, w_3, w_4). If $\deg w_2 = 4$, then there exists a vertex $x \neq v$ such that $x \sim \{w_1, w_2\}$. Besides, if $\deg w_3 = 4$, then there exists a vertex y, $y \notin \{v, x\}$ such that $y \sim \{w_3, w_4\}$.*

We are now ready to prove Theorem 3.

Proof. The proof is split into three parts.

Proof of Statement (i) of Theorem 3: Taking into account the conditions $\Delta(G) \leqslant 4$ and $\Delta(G) = \delta(G)$, we have to consider three possible cases.

Case 1. $\Delta(G) = 2$.

It is fairly easy to see that in this case G is a connected realization of K_2, and then it is obvious that $G \cong K_3 \cong C_3^2$.

Case 2. $\Delta(G) = 3$.

Since G is a 3-regular, locally connected graph, it follows that the neighborhood of any of its vertices v induces a connected graph $G(v)$ of order 3. There are only two such graphs: P_3 and K_3. Due to Lemma 2, the removal of all dominating vertices from graph $G(v)$ yields a graph with no isolated vertices. Graph P_3 does not possess this property, so that $G(v) \cong K_3$. This implies that G is a connected realization of graph K_3, and therefore, $G \cong K_4 \cong C_4^2$.

Case 3. $\Delta(G) = 4$.

Since G is a 4-regular, locally connected graph, it follows that the neighborhood of any of its vertices v induces a connected graph $G(v)$ of order 4. There are six such graphs: P_4, $K_{1,3}$, C_4, $K_{1,3} + e$, $K_4 - e$ and K_4. Due to Lemma 2, the removal of all dominating vertices from graph $G(v)$ yields a graph with no isolated vertices. This property does not hold for graphs $K_{1,3}$, $K_{1,3} + e$ and $K_4 - e$. Thus, graph $G(v)$ is isomorphic to K_4, or to C_4, or to P_4. We split our further consideration accordingly.

Case 3.1. $G(v) \cong K_4$.

In this case, $G[N(v) \cup \{v\}] \cong K_5$. Given that G is connected, this implies that $G \cong K_5 \cong C_5^2$.

Case 3.2. $G(v) \cong C_4$.

Since G is a regular graph, it follows that there exists a vertex $u \in V(G) \setminus (N(v) \cup \{v\})$ that is adjacent to at least one vertex of set $N(v)$. Let us partition set $N(v)$ into two subsets N_1 and N_2, where $N_1 = N(u) \cap N(v)$ and $N_2 = N(v) \setminus N_1$. It is clear that the set $X = N_1 \cup \{v\}$ is a maximal set in graph G, if $N_2 \neq \emptyset$. But then the subgraph G_X contains an isolated vertex, which contradicts Lemma 1. Thus, $N_2 = \emptyset$, i.e., $u \sim N(v)$. Since graph G is connected, we have $G \cong C_6^2$.

Case 3.3. $G(v) \cong P_4$.

None of the graphs C_5^2 or C_6^2 obtained above contains a vertex such that its neighborhood induces P_4. This implies that G is a connected realization of graph P_4. Depending on whether graph G is finite, we have two further cases to consider.

Case 3.3.1. Graph G is finite and $n = |G|$.

Let $v_1 \in V(G)$ be an arbitrary vertex. For graph G, denote the path P_4 induced by the set $N(v_1)$ by (v_3, v_2, v_n, v_{n-1}), where $v_3 \sim v_2 \sim v_n \sim v_{n-1}$. By Lemma 3, there exists a vertex $v_4 \neq v_1$ in graph G, such that $v_4 \sim \{v_2, v_3\}$, and a vertex $v_{n-2} \notin \{v_1, v_4\}$, such that $v_{n-2} \sim \{v_{n-1}, v_n\}$. Applying Lemma 3 to vertex v_2, we find a vertex $v_5 \neq v_2$, such that $v_5 \sim \{v_3, v_4\}$. If $v_5 = v_{n-2}$ (which happens for $n = 7$), then $v_4 \sim v_{n-1}$; otherwise, graph $G(v_{n-2})$ is not connected and $G \cong C_7^2$. If $v_5 \neq v_{n-2}$, then apply Lemma 3 to vertex v_3.

We now describe the i-th step of the process, where $1 \leqslant i \leqslant n - 6$ and $n \geqslant 7$. Below, for a vertex denoted by $*$ we have $G(*) \cong P_4$. Apply Lemma 3 to a vertex v_{i+1}, such that its neighborhood induces a simple path of the form $(*, *, v_{i+2}, v_{i+3})$ and find a vertex v_{i+4}, such that $v_{i+4} \sim \{v_{i+2}, v_{i+3}\}$. As a result, the degree of vertex v_{i+2} (or vertex v_{i+3}) becomes equal to 4 (to 3, respectively). For $i = n - 6$, a new vertex v_{i+4} coincides with v_{n-2}, and, therefore, there exists the edge that connects the vertices $v_{i+3} = v_{n-3}$ and v_{n-1}; otherwise, graph $G(v_{n-2})$ is not connected. It can be seen that the described process leads to graph C_n^2.

Case 3.3.2. Graph G is infinite.

Let v_0 be an arbitrary vertex of graph G. For graph G, denote the path P_4 induced by the set $N(v_0)$ by $(v_{-2}, v_{-1}, v_1, v_2)$, where $v_{-2} \sim v_{-1} \sim v_1 \sim v_2$. Due to Lemma 3, there exists a vertex $v_{-3} \neq v_0$, such that $v_{-3} \sim \{v_{-2}, v_{-1}\}$, and a vertex $v_3 \notin \{v_{-3}, v_0\}$, such that $v_3 \sim \{v_1, v_2\}$. This implies that $\deg v_{-1} = \deg v_1 = 4$ and $G(v_{-1}) \cong G(v_1) \cong P_4$. Furthermore, we have that $v_{-3} \not\sim \{v_2, v_3\}$ and $v_3 \not\sim \{v_{-3}, v_{-2}\}$.

We now describe the i-th step of the process, $i \geqslant 1$. Apply Lemma 3 to vertices v_{-i} and v_i, such that their neighborhoods induce simple paths of the form $(v_{-i-2}, v_{-i-1}, *, *)$ and $(*, *, v_{i+1}, v_{i+2})$, respectively. Find vertices $v_{-i-3} \neq v_{-i}$ and $v_{i+3} \notin \{v_{-i-3}, v_i\}$, such that $v_{-i-3} \sim \{v_{-i-2}, v_{-i-1}\}$ and $v_{i+3} \sim \{v_{i+1}, v_{i+2}\}$. Then the degree of vertex v_{-i-1} (or that of vertex v_{i+1}) becomes equal to 4, and graph $G(v_{-i-1})$ (or graph $G(v_{i+1})$, respectively) becomes isomorphic to P_4. It is also clear that $v_{-i-3} \not\sim \{v_{i+2}, v_{i+3}\}$ and $v_{i+3} \not\sim \{v_{-i-3}, v_{-i-2}\}$; otherwise, graph G is not connected. It is fairly easy to verify that the described process leads to graph $P^2_{\infty,\infty}$.

We have considered all cases. Statement (i) is proved.

Proof of Statement (ii) of Theorem 3: We need to consider the following cases.

Case 1. $\Delta(G) = 3$ and $\delta(G) = 2$.

Since $\delta(G) = 2$, if follows that $V(G)$ contains a vertex of degree 2. Let v be such a vertex, and $N(v) = \{v_1, v_2\}$. Since graph $G(v)$ is connected, we deduce that $v_1 \sim v_2$. The condition $\Delta(G) = 3$ implies that in graph G either vertex v_1 or vertex v_2 is of degree 3. Due to symmetry, we may assume that $\deg v_1 = 3$. Let u be a neighbor of vertex v_1 different from v and v_2. Then $u \sim v_2$, and, therefore, $\deg v_2 = 3$. Since graph $G(u)$ is connected, it follows that $N(u) = \{v_1, v_2\}$. Taking into account connectivity of graph G, we derive $G \cong P^2_4$.

Case 2. $\Delta(G) = 4$ and $\delta(G) = 3$.

In this case graph G contains a vertex of degree 3. Let v be such a vertex and $N(v) = \{v_1, v_2, v_3\}$. Our further consideration is split into two subcases.

Case 2.1. $v_1 \sim v_2 \sim v_3 \sim v_1$, i.e., $G(v) \cong K_3$.

Since $\Delta(G) = 4$, thee exists a vertex $u \in V(G) \setminus (N(v) \cup \{v\})$ such that $N(u) \cap N(v) \neq \emptyset$. Without loss of generality, assume that $u \sim v_1$. Taking into account symmetry and connectivity of graph $G(v_1)$, derive $u \sim v_2$. On the other hand, since $G(u)$ is connected and $\delta(G) = 3$, we have that $u \sim v_3$. Thus, $N(u) = \{v_1, v_2, v_3\}$. Since graph G is connected, we have $G \cong K_5 - e$.

Case 2.2. $v_1 \sim v_2 \sim v_3$, i.e., $G(v) \cong P_3$.

Notice that the equality $\deg v_2 = 3$ is impossible; otherwise, graphs $G(v_1)$ and $G(v_3)$ would not be connected. Thus, there exists a vertex $u \in V(G) \setminus (N(v) \cup \{v\})$ such that $u \sim v_2$. Since each graph $G(v_i)$, $1 \leqslant i \leqslant 3$, is connected, we deduce $u \sim \{v_1, v_3\}$. Thus, $G[N(v) \cup \{u, v\}] \cong W_4$. Graph W_4 satisfies all conditions of this statement; therefore, for $V(G) = N(v) \cup \{u, v\}$ we may conclude that $G \cong W_4$. Otherwise, graph G contains another vertex, say, vertex w that is adjacent to at least one of the vertices u, v_1, v_3. For certainty, assume that $w \sim u$ (the cases that $w \sim v_i$, where $i \in \{1, 3\}$, are analogous). Due to connectivity of graphs $G(u)$ and $G(w)$, as well as by the equality $\delta(G) = 3$, we conclude that w is also adjacent to both v_1 and v_3. But then $\deg w = 3$ and $G[N(v) \cup \{u, v, w\}] \cong O_2 + P_4$. Due to its connectivity, graph G has no other vertices and $G \cong O_2 + P_4$.

This concludes the proof of Statement (ii).

Proof of Statement (iii) of Theorem 3: Since G is a locally connected graph, it follows that $\delta(G) \geqslant 2$. This and the relation $\Delta(G) - \delta(G) = 2$ imply $\Delta(G) \geqslant 4$. Since $\Delta(G) \leqslant 4$

by condition, we obtain $\Delta(G) = 4$ and, in turn, $\delta(G) = 2$. Thus, $V(G)$ contains vertices of degree 2 and of degree 4, and, possibly, those of degree 3. Let v denote a vertex of degree 4, and further denote $N(v) = \{v_1, v_2, v_3, v_4\}$. Then $G(v)$ is a connected graph of order 4. There are exactly six such graphs: P_4, $K_{1,3}$, C_4, $K_{1,3} + e$, $K_4 - e$, K_4. It is fairly easy to see that $G(v)$ is isomorphic neither to $K_4 - e$ nor to K_4. The remainder of this proof considers the other four options for graph $G(v)$.

Case 1. $v_1 \sim v_i$ for each $i \in \{2, 3, 4\}$, i.e., $G(v) \cong K_{1,3}$.

In this case $\deg v_1 = 4$, and therefore, the degree of each vertex v_i for $i \in \{2, 3, 4\}$ is equal to 2; otherwise, a graph $G(v_i)$ would not be connected. Due to its connectivity, graph G has no other vertices and $G \cong K_{1,1,3}$.

Case 2. $v_1 \sim v_i$ for each $i \in \{2, 3, 4\}$ and $v_2 \sim v_3$, i.e., $G(v) \cong K_{1,3} + e$.

In this case $\deg v_1 = 4$, and therefore, $\deg v_4 = 2$; otherwise, graph $G(v_4)$ would not be connected. If $V(G) = N(v) \cup \{v\}$ then we derive $G \cong K_2 + \overline{P_3}$. Otherwise, $V(G)$ contains another vertex, say, vertex u that is adjacent to at least one of the vertices v_2 or v_3. By symmetry, we may assume that $u \sim v_2$. This, due to connectivity of graph $G(v_2)$, implies $u \sim v_3$. But in this case $\deg u = 2$ and $G[N(v) \cup \{u, v\}] \cong H_1$. Taking into account connectivity of graph G, we derive $G \cong H_1$.

Case 3. $v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1$, i.e., $G(v) \cong C_4$.

Since graph G is connected, the condition $\delta(G) = 2$ implies that at least one vertex in $N(v)$ is of degree 4 in graph G. Without loss of generality, we may assume that $\deg v_1 = 4$. Let u be a neighbor of v_1 that does not belong to the set $\{v, v_2, v_4\}$. Due to connectivity of graph $G(v_1)$, we deduce that vertex u is adjacent to at least one of the vertices v_2 or v_4. Notice that the case that $u \sim \{v_2, v_4\}$ is impossible, since then either $G(u)$ or graph $G(v_3)$ would not be connected. By symmetry, we may assume that $u \sim v_2$ and $u \not\sim \{v_3, v_4\}$. But then it is obvious that the degree of vertex u in graph G is equal to 2, and $G[N(v) \cup \{u, v\}] \cong H_2$. Graph H_2 satisfies all conditions of this statement. Thus, if $V(G) = N(v) \cup \{u, v\}$, then $G \cong H_2$. Otherwise, $V(G)$ contains another vertex, say, vertex w such that $w \notin \{u, v, v_1, v_2\}$ and $w \sim \{v_3, v_4\}$. Then $\deg w = 2$ and $G[N(v) \cup \{u, v, w\}] \cong H_3$. Due to its connectivity, graph G has no other vertices and $G \cong H_3$.

Thus, we have considered the cases that graph $G(v)$ is isomorphic to one of the graphs $K_{1,3}$, $K_{1,3} + e$ or C_4. We may therefore assume that in all cases to be considered in the remainder of this proof the neighborhood of each vertex of degree 4 induces a path P_4 in graph G. It is fairly easy to see that in this case no vertex in G has the neighborhood that induces K_3. Thus, we are left to consider two cases with respect to $\boldsymbol{N}(G)$.

Case 4. $\boldsymbol{N}(G) = \{K_2, P_4\}$.

Let u be a vertex of degree 2 in graph G and let $N(u) = \{u_1, u_2\}$, $u_1 \sim u_2$. The condition $\Delta(G) = 4$ implies that for at least one of the vertices u_1 or u_2 its neighborhood in G induces P_4. By symmetry, we may assume that $G(u_1) \cong P_4$. Then there exist vertices u_3 and u_4 that are adjacent to u_1 and such that $u_2 \sim u_3 \sim u_4$. Since graph G has no vertices of degree 3, it follows that $V(G)$ contains one more vertex, say, vertex w, adjacent to u_2. Then taking into account that $G(u_2) \cong P_4$, we deduce that $w \sim u_3$. Vertices w and

u_4 are not adjacent; otherwise, the neighborhood $N(u_3)$ would induce a cycle C_4. Thus, $G(w) \cong G(u_4) \cong K_2$. Due to its connectivity, graph G has no other vertices and $G \cong H_4$.

Case 5. $\boldsymbol{N}(G) = \{K_2, P_3, P_4\}$.

Let v_2 be an arbitrary vertex of degree 3 in graph G, and let $N(v_2) = \{v_1, v_3, v_4\}$, $v_1 \sim v_3 \sim v_4$. Then it follows that $\deg v_3 = 4$; otherwise, either graph $G(v_1)$ or $G(v_4)$ is not connected. Thus, there exists a vertex $v_5 \in V(G) \setminus (N(v_2) \cup \{v_2\})$ such that $v_3 \sim v_5$. Since $G(v_3) \cong P_4$, by symmetry we may assume that $v_4 \sim v_5$. It is clear that $v_1 \not\sim v_5$, since otherwise, $G(v_3) \cong C_4$. Besides, $\deg v_1 = 2$. Therefore, for $n = |G| = 5$ we have that $G \cong P_5^2$. If $|G| = 6$, apply Lemma 4 to vertex v_3 and find a vertex $v_6 \neq v_3$ such that $v_6 \sim \{v_4, v_5\}$. In this case, $G \cong P_6^2$. If $|G| \geqslant 7$, then apply Lemma 4 to vertex v_4.

We now describe the i-th step of the process. Apply Lemma 4 to vertex v_{i+2} the neighborhood of which induces a path P_4 of the form $(*, *, v_{i+3}, v_{i+4})$, and find a vertex v_{i+5} such that $v_{i+5} \sim \{v_{i+3}, v_{i+4}\}$. As a result, the degree of vertex v_{i+3} (or degree of vertex v_{i+4}) becomes equal to 4 (or to 3, respectively). If G is a finite graph and $n = |G| \geqslant 6$, then for $i = n - 5$ the new vertex v_{i+5} will coincide with v_n, and then obviously $G(v_{n-1}) \cong P_3$ and $G(v_n) \cong K_2$. If G is an infinite graph, them apply Lemma 4 to vertex v_{i+3}. It is easy to see that the described process leads to graph P_n^2 (or to graph $P_{1,\infty}^2$, respectively).

This completes the proof of Statement (iii), and therefore Theorem 3 is completely proved. □

3. Sufficient conditions of hamiltonicity of locally connected graphs

In this section we present new sufficient local conditions of hamiltonicity, more general than those previously known. Throughout this section we deal with finite graphs.

Remind that a simple cycle C' is called an *extension* of a simple cycle C in a graph G, if $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. An extension C' of a cycle C is denoted by $C \to C'$.

For the sake of simplicity, a subgraph of G induced by a vertex set $\{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ is denoted by $G[u_1, u_2, \ldots, u_k]$ instead of $G[\{u_1, u_2, \ldots, u_k\}]$.

Before we proceed, we first state the following simple properties of locally connected graphs.

Property 1. *Each edge of a locally connected graph is contained in a triangle.*

Property 2. *If G is a locally connected graph and an edge $uv \in E(G)$ is contained in the unique triangle $G[u, v, w]$, then the edges uw and vw are contained in at least two triangles of graph G if and only if $\deg u \geqslant 3$, $\deg v \geqslant 3$.*

Property 3. *A connected, locally connected graph is 2-connected.*

The following theorem is due to Kikust [18], see also [29, p. 106].

Theorem 4 ([18])**.** *Each connected, locally connected* 5*-regular graph is hamiltonian.*

Theorem 4 admits a natural generalization in the form of Theorem 2 given in Introduction and proved by Hendry [14]. Taking Theorems 2 and 3, we may pose the following question: *Is it true that each connected, locally connected graph with* $\Delta(G) = 5$ *and* $\delta(G) \geqslant 3$ *is hamiltonian?* Graph $K_{1,1,4}$ demonstrates that a stronger inequality $\delta(G) \geqslant 2$ is not acceptable. The answer to the question we have just posed comes from the following statement.

Theorem 5. *Let* G *be a connected, locally connected graph with* $\Delta(G) = 5$ *and* $\delta(G) \geqslant 3$*. Then* G *is fully cycle extendable.*

Proof. Suppose that G is a graph that satisfies the conditions of the theorem and such that G contains a nonextendable non-Hamilton cycle C. Notice that due to Property 3 graph G is 2-connected. Let $V(C) = \{v_1, v_2, \dots, v_p\}$, where vertices are numbered in the order of traversing the cycle. By assumption, the set $S = V(G) \setminus V(C)$ is not empty, so that vertex v_1 is adjacent to some vertex in S; in the remainder of this proof, we denote such a vertex of S by x. We now prove several useful properties of graph G. In what follows, the subscripts of the vertices in C are taken modulo p.

Claim 1. *If* $v_i \in V(C)$ *and* $z \in N(v_i) \cap S$*, then* $z \not\sim \{v_{i-1}, v_{i+1}\}$*.*

Proof. Assuming the opposite, we would have

$$C \to (z, v_i, v_{i+1}, v_{i+2}, \dots, v_{i-2}, v_{i-1}, z), \text{ if } z \sim v_{i-1},$$
$$C \to (z, v_{i+1}, v_{i+2}, \dots, v_{i-2}, v_{i-1}, v_i, z), \text{ if } z \sim v_{i+1}.$$

however, this contradicts the choice of cycle C. Claim 1 is proved. □

Claim 2. *For vertex* v_1*, the relation* $\deg v_1 \neq 3$ *holds.*

Proof. Due to Claim 1, vertex x is adjacent to neither vertex v_2 nor vertex v_p. Therefore, if $\deg v_1 = 3$, then graph $G(v_1)$ contains an isolated vertex x. This contradicts the fact that G is a locally connected graph. Thus, the equality $\deg v_1 = 3$ is impossible. Claim 2 is proved. □

Claim 3. *For vertex* v_1*, the relation* $\deg v_1 \neq 4$ *holds.*

Proof. Suppose that $\deg v_1 = 4$. Due to Property 1, the edge xv_1 is contained in some triangle of graph G, say, in the triangle $T_1 = G[x, y, v_1]$. It follows that $y \in V(C)$. Indeed, if $y \in S$, then applying Claim 1, we would have $\{x, y\} \not\sim \{v_2, v_p\}$, which contradicts to connectivity of graph $G(v_1)$.

Let $y = v_i$. Then, as shown above, $3 \leqslant i \leqslant p - 1$. We show that for $i \in \{3, p-1\}$ graph G contains an extension of cycle C. Due to symmetry, without loss of generality, assume that $i = 3$. Notice that $v_2 \not\sim v_p$, since otherwise $C \to (x, v_3, v_4, \dots, v_{p-1}, v_p, v_2, v_1, x)$. On the other hand, since graph $G(v_1)$ is connected, it follows that $v_3 \sim v_p$ and, therefore, $G[x, v_1, v_2, v_3, v_p] \cong K_{1,1,3}$. This and the choice of C imply that $p \geqslant 5$. Since by construction $T_2 = G[v_1, v_2, v_3]$ is the only triangle that contains the edge v_1v_2, due to

Property 2 we deduce that graph G contains another triangle T_3 such that $T_3 \neq T_2$ and $v_2v_3 \in E(T_3)$. Actually, the only possibility is that $T_3 = G[v_2, v_3, v_4]$. This yields $C \rightarrow (x, v_3, v_2, v_4, \ldots, v_{p-1}, v_p, v_1, x)$, a contradiction, which eventually means that $i \notin \{3, p-1\}$.

Thus, we have proved that $4 \leqslant i \leqslant p-2$. Now, due to connectivity of graph $G(v_1)$ and the relation $\Delta(G) = 5$, by symmetry we deduce $v_i \sim v_2 \sim v_p$, which is a contradiction again. Indeed, taking into account what has been proved above, the triangle $T_1 = G[x, v_1, v_i]$ is the only one that contains the edge xv_1. Thus, by Property 2, there exists a triangle T_4 such that $T_4 \neq T_1$ and $xv_i \in E(T_4)$. It is clear that there are exactly two possibilities: either $T_4 = G[x, v_{i-1}, v_i]$ or $T_4 = G[x, v_i, v_{i+1}]$. This, however, contradicts Claim 1, which guarantees that $x \not\sim \{v_{i-1}, v_{i+1}\}$. Claim 3 is proved. □

Combining Claims 2 and 3 with $\Delta(G) = 5$, we derive that $\deg v_1 = 5$. This relation is used in the remainder of the proof of the theorem.

Claim 4. *If $W = N(v_1) \setminus \{x\}$, then $W \subseteq V(C)$.*

Proof. Assume the opposite, i.e., that $W \cap S \neq \emptyset$. Then due to Claim 1 and connectivity of graph $G(v_1)$, we immediately deduce that $|W \cap S| = 1$. As before, assume that the edge xv_1 is contained in the triangle $G[x, y, v_1]$. Let z be the fifth vertex that is adjacent to v_1 (along with v_2, v_p, x, y). Depending on the locations of vertices y and z two cases are possible.

Case 1. $y \in S$, $z \in V(C)$.

Let $z = v_i$. Applying reasoning similar to that used in the proof of Claim 3, we derive that $4 \leqslant i \leqslant p-2$. Since $\Delta(G) = 5$ and graph $G(v_1)$ is connected, by symmetry we have that $x \sim v_i \sim v_2 \sim v_p$ and, therefore, $G(v_1) \cong P_5$. On the other hand, due to connectivity of graph $G(v_i)$, we deduce that vertex v_{i+1} must be adjacent to at least one of the vertices v_2 or v_{i-1}. In turn, this implies that

$$C \rightarrow (x, v_i, v_{i-1}, \ldots, v_3, v_2, v_{i+1}, \ldots, v_{p-1}, v_p, v_1, x), \text{ if } v_{i+1} \sim v_2,$$
$$C \rightarrow (x, v_i, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{p-1}, v_p, v_1, x), \text{ if } v_{i+1} \sim v_{i-1}.$$

Thus, vertex v_{i+1} is isolated in graph $G(v_i)$, a contradiction that closes our consideration of Case 1.

Case 2. $y \in V(C)$, $z \in S$.

Let $y = v_i$. Then, if $x \sim z$ we can establish a contradiction to connectivity of graph $G(v_i)$ in a similar way we have done it in the proof of Case 1. Thus, we may assume that $x \not\sim z$. Due to Property 1, the edge zv_1 is contained in some triangle of graph G. This, as well as the fact that $z \not\sim \{x, v_2, v_p\}$ imply that such a triangle could be only the triangle $G[z, v_1, v_i]$. By symmetry and connectivity of graph $G(v_1)$, without loss of generality, we may assume that $v_2 \sim v_i$. This and the equality $\Delta(G) = 5$ imply that $i = 3$. Notice that $v_2 \not\sim v_p$; otherwise, we would have $C \rightarrow (x, v_3, v_4, \ldots, v_{p-1}, v_p, v_2, v_1, x)$, a contradiction. Thus, since graph $G(v_1)$ is connected, the only possibility is that $v_3 \sim v_p$; in turn, this together with the equality $\Delta(G) = 5$ leads to $|V(C)| = p = 4$; besides, $G[x, z, v_1, v_2, v_3, v_4] \cong K_{1,1,4}$. This and the condition $\delta(G) \geqslant 3$ finally yield that graph $G(x)$ is not connected. This contradiction completes our treatment of Case 2, and therefore, Claim 4 is proved. □

Claim 5. *There exist vertices $v_i, v_j \in V(C)$, where $3 \leqslant i \leqslant j-2$ and $j \leqslant p-1$, such that edge xv_1 is contained in exactly two triangles $T_1 = G[x, v_1, v_i]$ and $T_2 = G[x, v_1, v_j]$.*

Proof. Let the edge xv_1 is contained in a triangle $T_1 = G[x, y, v_1]$ of graph G. By Claim 4, we deduce that $y \in V(C)$, e.g., $y = v_i$. This and Claim 1 imply $3 \leqslant i \leqslant p-1$. Further, due to Property 2, there exists a triangle T_2, $T_2 \neq T_1$, such that either $xv_1 \in E(T_2)$ or $xv_i \in E(T_2)$. We show that the latter option reduces to the former one. Indeed, assume that $xv_i \in E(T_2)$ and $T_2 = G[x, z, v_i]$. Using the same argument as in the proof of Claim 4, we derive that $z \in V(C)$ and, say, $z = v_k$. Without loss of generality, we may assume that $i < k$. Besides, $i \leqslant k-2$ and $k \leqslant p-1$ due to Claim 1. Define a bijective mapping $\varphi : V(C) \cup \{x\} \to V(C) \cup \{x\}$ as $\varphi(x) = x$ and

$$\varphi(v_t) = \begin{cases} v_{i+1-t}, & \text{if } 1 \leqslant t \leqslant i, \\ v_{p+i+1-t}, & \text{if } i+1 \leqslant t \leqslant p. \end{cases}$$

Renumbering the vertices of the set $V(C)$ in accordance with φ, we obtain $G[\varphi(V(T_1))] = G[x, v_1, v_i]$ and $G[\varphi(V(T_2))] = G[x, v_1, \varphi(v_k)]$, where $\varphi(v_k) = v_j$ and $j = p+i+1-k$. Thus $xv_1 \in E(T_2)$ and $i+2 \leqslant j \leqslant p-1$. Now we may conclude that $T_2 = G[x, v_1, v_j]$ and $3 \leqslant i \leqslant j-2$, $j \leqslant p-1$. Claim 5 is proved. □

Claim 6. *Claim 5 holds for vertices v_i and v_j, where $i = 3$ and $j = p-1$.*

Proof. Assuming the contrary, by symmetry we have one of the following two cases: $4 \leqslant i \leqslant j-2$, $j = p-1$ and $4 \leqslant i \leqslant j-2$, $j \leqslant p-2$. In both cases we will arrive at a contradiction.

Case 1. $4 \leqslant i \leqslant j-2$ and $j = p-1$.

First, we show that $v_2 \not\sim v_i$. If the opposite is true, then $v_{i-1} \not\sim v_{i+1}$, since otherwise

$$C \to (x, v_i, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{p-1}, v_p, v_1, x),$$

which contradicts to the choice of the cycle C. Similarly, if $v_2 \sim v_{i+1}$, then

$$C \to (x, v_i, v_{i-1}, \ldots, v_3, v_2, v_{i+1}, \ldots, v_{p-1}, v_p, v_1, x),$$

which is a contradiction. Thus, vertex v_2 is not adjacent to vertex v_{i+1}. In turn, the condition $\Delta(G) = 5$ together with Claim 1 imply that $v_1 \not\sim v_{i+1}$ and $x \not\sim v_{i+1}$, respectively. But then vertex v_{i+1} is isolated in graph $G(v_i)$. The obtained contradiction proves that $v_2 \not\sim v_i$. The relation $v_i \not\sim v_p$ can be proved analogously. Besides, $v_2 \not\sim v_p$, since, otherwise, we would have $C \to (x, v_1, v_p, v_2, v_3, \ldots, v_{p-2}, v_{p-1}, x)$, which is a contradiction. Now, taking into account Claim 1 and connectivity of $G(v_1)$ we deduce $v_2 \sim v_{p-1}$ and, therefore, $\deg v_{p-1} = 5$. This and the above part of the proof imply that the triangle $T_1 = G[v_1, v_{p-1}, v_p]$ is the only one that contains $v_1 v_p$. Applying Property 2 to triangle T_1 we conclude that there exists a triangle T_2, $T_2 \neq T_1$, that contains the edge $v_{p-1} v_p$. It is clear that the only possibility for T_2 is $T_2 = G[v_{p-2}, v_{p-1}, v_p]$. But then $C \to (x, v_1, v_2, v_3, \ldots, v_{p-2}, v_p, v_{p-1}, x)$, a contradiction.

Case 2. $4 \leqslant i \leqslant j-2$ and $j \leqslant p-2$.

As in the proof of Case 1, we can derive that $\{v_2, v_p\} \not\sim \{v_i, v_j\}$. This together with $x \not\sim \{v_2, v_p\}$ implies that $G(v_1)$ is not connected, a contradiction.

Thus, in either case we have arrived at a contradiction, and we therefore conclude that $i = 3$ and $j = p - 1$. Claim 6 is proved. □

Claim 7. *The degree of vertex x is equal to* 3*, while the length of cycle C is at least* 7*, i.e., $\deg x = 3$ and $p \geqslant 7$. Additionally, $G(x) = P_3 = (v_{p-1}, v_1, v_3)$ and $G(v_1) = P_5 = (v_2, v_3, x, v_{p-1}, v_p)$, where the vertices are listed in the order of traversing the corresponding paths.*

Proof. Assume that $\deg x \neq 3$. Since $\Delta(G) = 5$ and $\delta(G) \geqslant 3$, then either $\deg x = 4$ or $\deg x = 5$. We present the proof for $\deg x = 4$; the case that $\deg x = 5$ is analogous.

Due to Claims 3 and 6, the edge xv_1 belongs to exactly two triangles $G[x, v_1, v_3]$ and $G[x, v_1, v_{p-1}]$. Thus, if y is the forth vertex adjacent to x, then due to connectivity of graph $G(x)$ at least one of the two following statements is valid: either $y \sim v_3$ or $y \sim v_{p-1}$. Due to symmetry, we may assume that $y \sim v_3$. In turn, this and Claim 1 (applied to vertex v_3) imply that $y \in V(C)$.

Now, let $y = v_s$, where s is the number of the corresponding vertex of $V(C)$. As proved above, $5 \leqslant s \leqslant p - 3$. Assume that s satisfies the inequalities $6 \leqslant s \leqslant p - 4$. Since graph $G(x)$ is connected, then at least one of the two possible edges v_3v_s or v_sv_{p-1} is indeed present in G. E.g., let $v_3 \sim v_s$ (due to symmetry, the case that $v_s \sim v_{p-1}$ does not require additional consideration). It is fairly easy to see that $v_2 \not\sim v_4$. This and connectivity of graph $G(v_3)$ imply that $v_4 \sim v_s$. We show that v_{s+1} is an isolated vertex in $G(v_s)$. Indeed, if $v_{s-1} \sim v_{s+1}$, then

$$C \to (x, v_s, v_4, v_5, \dots, v_{s-1}, v_{s+1}, v_{s+2}, \dots, v_{p-2}, v_{p-1}, v_p, v_1, v_2, v_3, x).$$

On the other hand, if $v_4 \sim v_{s+1}$, then

$$C \to (x, v_s, v_{s-1}, \dots, v_5, v_4, v_{s+1}, v_{s+2}, \dots, v_{p-2}, v_{p-1}, v_p, v_1, v_2, v_3, x).$$

Thus, $v_{s+1} \not\sim \{v_4, v_{s-1}\}$. Notice that $v_{s+1} \not\sim \{x, v_3\}$ and conclude that v_{s+1} is isolated in $G(v_s)$; a contradiction. Thus, for s only two options are left: either $s = 5$ or $s = p - 3$. Due to symmetry, we may assume that $s = 5$. Since graph $G(x)$ is connected, it follows that vertex v_5 is adjacent to at least one of the vertices v_3 or v_{p-1}. Below it is assumed that $v_3 \sim v_5$; the case that $v_5 \sim v_{p-1}$ can be considered analogously.

We now turn to vertex v_2. It is clear that $v_2 \not\sim \{x, v_4, v_p\}$. On the other hand, since $\delta(G) \geqslant 3$, it follows that v_2 is adjacent to at least one of the vertices v_5 or v_{p-1}; by symmetry, without loss of generality, we may assume that $v_2 \sim v_5$. This together with connectivity of graph $G(v_5)$ and the fact that $x \not\sim v_6$ imply that either $v_2 \sim v_6$ or $v_4 \sim v_6$. But then

$$C \to (x, v_5, v_4, v_3, v_2, v_6, \dots, v_{p-2}, v_{p-1}, v_p, v_1, x), \text{ if } v_2 \sim v_6,$$
$$C \to (x, v_5, v_4, v_6, \dots, v_{p-2}, v_{p-1}, v_p, v_1, v_2, v_3, x), \text{ if } v_4 \sim v_6,$$

which contradicts to the choice of cycle C. Thus, $v_2 \not\sim \{v_5, v_{p-1}\}$. But this in turn contradicts connectivity of $G(v_2)$. Therefore, $\deg x \neq 4$. Thus, we deduce that $\deg x = 3$.

According to Claim 6, $p \geqslant 5$. If $p = 5$, then $C \to (x, v_3, v_2, v_1, v_5, v_4)$. If $p = 6$, then from the condition $\delta(G) \geqslant 3$, we have that v_4 is adjacent to a vertex on C (by Claim 1). By symmetry, we derive that $v_2 \sim v_4$. This gives $C \to (x, v_3, v_4, v_2, v_1, v_6, v_5, x)$, a contradiction that proves the inequality $p \geqslant 7$.

Besides, a simple argument shows that for graph G the existence of at least one of the edges v_2v_{p-1}, v_2v_p, v_3v_{p-1} or v_3v_p contradicts either to local connectivity of graph G, or to the choice of cycle C. Indeed, suppose, e.g., that $v_2 \sim v_{p-1}$. Then, since graph $G(v_{p-1})$ is connected, at least one of the two possible edges v_2v_{p-2} or $v_{p-2}v_p$ is present in graph G. From this we deduce

$$C \to (x, v_1, v_p, v_{p-1}, v_2, v_{p-2}, v_{p-3}, \dots, v_4, v_3, x), \text{ if } v_2 \sim v_{p-2},$$
$$C \to (x, v_1, v_2, v_3, v_4, \dots, v_{p-3}, v_{p-2}, v_p, v_{p-1}, x), \text{ if } v_{p-2} \sim v_p.$$

These contradictions demonstrate that $v_2 \not\sim v_{p-1}$. In turn, this implies that $G(x) = P_3 = (v_{p-1}, v_1, v_3)$ and $G(v_1) = P_5 = (v_2, v_3, x, v_{p-1}, v_p)$, which completes the proof of Claim 7. □

Denote $l = \lceil p/2 \rceil - 3$. Then Claim 7 implies that $l \geqslant 1$.

Claim 8. *For each integer k, $1 \leqslant k \leqslant l$, the following holds:*

(i) $\{v_{2k}v_{2k+3}, v_{2k+1}v_{2k+3}\} \subset E(G)$;
(ii) $\deg v_{2k} = 3$, $G(v_{2k}) = P_3 = (v_{2k-1}, v_{2k+1}, v_{2k+3})$ *and* $\deg v_{2k+1} = 5$, $G(v_{2k+1}) = P_5 = (v_{2k-2}, v_{2k-1}, v_{2k}, v_{2k+3}, v_{2k+2})$, *where* $v_{2k-2} = x$ *for* $k = 1$.

Proof. Figure 2 illustrates Claim 8 for the cases of odd ($p = 15$) or even ($p = 16$) number of vertices in C. In this proof we will use the structural properties of G established in Claims 5, 6 and 7 sometimes without explicit references.

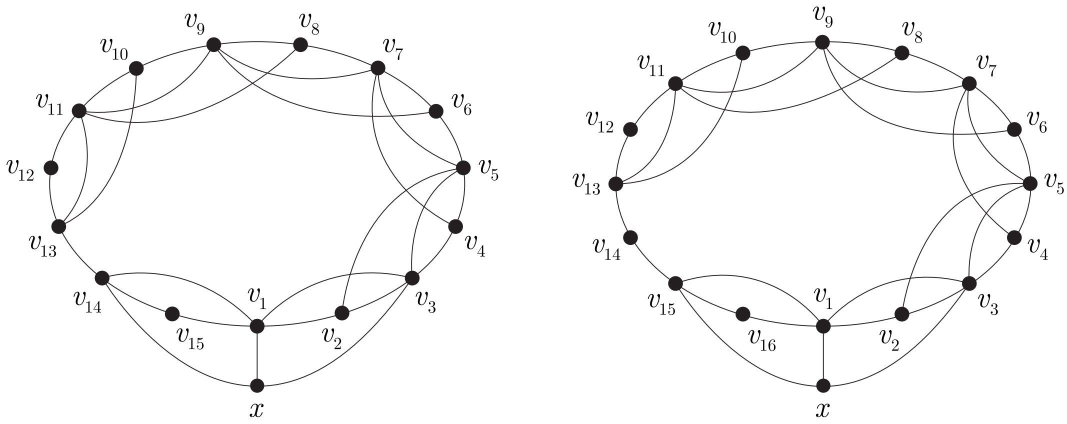

Fig. 2. An illustration to Claim 8

First, we prove (i) and (ii) for $k = 1$. Let us show that the relation $\{v_2v_5, v_3v_5\} \subset E(G)$ holds. Let T_1 denote the triangle $G[v_1, v_2, v_3]$. By Property 2, graph G contains a triangle T_2 such that $T_2 \neq T_1$ and either $v_1v_2 \in E(T_2)$ or $v_2v_3 \in E(T_2)$. If $v_1v_2 \in E(T_2)$, then either $T_2 = G[v_1, v_2, v_{p-1}]$ or $T_2 = G[v_1, v_2, v_p]$; however neither of these is possible, since due to Claim 7 we have that $G(v_1) = P_5 = (v_2, v_3, x, v_{p-1}, v_p)$ and, therefore, $v_2 \not\sim \{v_{p-1}, v_p\}$. Thus the only possibility is $v_2v_3 \in E(T_2)$.

Let $T_2 = G[y, v_2, v_3]$. Then by Claim 1, we deduce that $y \in V(C)$; e.g., assume that $y = v_s$. It is easy to verify that $s = 5$; otherwise as in the proof of Claim 7 we would come to a contradiction. Thus, we have proved that $\{v_2v_5, v_3v_5\} \subset E(G)$, i.e., (i) holds.

If $v_2 \sim v_4$, we have $C \to (x, v_3, v_2, v_4, v_5, \dots, v_p, v_1, x)$. Thus, $v_2 \not\sim v_4$ and we deduce that $G(v_3) = P_5 = (x, v_1, v_2, v_5, v_4)$ and hence $\deg v_3 = 5$. We now show that $\deg v_2 = 3$. Assuming the opposite, we would find a vertex z adjacent to v_2 such that due to connectivity of graph $G(v_2)$ we would have $z \sim v_5$. Notice that $z \in V(C)$; otherwise, graph $G(v_5)$ would not be connected. Let $z = v_t$, where $7 \leqslant t \leqslant p-3$ (the cases that $t = 6$ or $t = p-2$ lead to a contradiction to the choice of cycle C). Since $G(v_5)$ is connected and $v_6 \not\sim \{v_2, v_4\}$, it follows that the only available option is that $v_6 \sim v_t$. This and connectivity of graph $G(v_4)$ imply that $v_4 \sim v_t$ and, therefore, taking into account the equality $\Delta(G) = 5$, we derive that $t = 7$. Since $G(v_7)$ is connected, then at least one of the edges v_2v_8, v_4v_8 or v_6v_8 must belong to graph G. In any case any condition $v_2 \sim v_8$, $v_4 \sim v_8$ or $v_6 \sim v_8$ contradicts the choice of cycle C. But then vertex v_8 is isolated in $G(v_7)$; a contradiction again. Thereby, we have established that vertex z does not exist, i.e., $\deg v_2 = 3$, $G(v_2) = P_3 = (v_1, v_3, v_5)$ and (ii) holds. Thus, Claim 8 for $k = 1$ is proved.

The remaining part of the proof is by induction. Assume that the claim holds for $k-1$, where $1 \leqslant k-1 < l$. We will prove it also holds for k. Considering the triangle $T_{2k-1} = G[v_{2k-1}, v_{2k}, v_{2k+1}]$ due to Property 2 we notice that the edge $v_{2k}v_{2k+1}$ belongs to some triangle T_{2k}, where $T_{2k} \neq T_{2k-1}$; say, $T_{2k} = G[z, v_{2k}, v_{2k+1}]$. Claim 1 implies $z \in V(C)$; e.g. we may assume that $z = v_r$. It is fairly easy to see that $r \neq p-1$. Furthermore, the assumption that $r \in \{2k+2, p-2, p\}$ leads to a contradiction to the choice of cycle C: each time the cycle can be extended. Thus, $2k+3 \leqslant r \leqslant p-3$. We will show that $r = 2k+3$. To prove that, assume the opposite, i.e., $2k+3 < r \leqslant p-3$. Since for $v_{2k} \sim v_{2k+2}$ cycle C can be extended to the cycle

$$(x, v_3, v_2, v_5, v_4, v_7, v_6, \dots, v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, v_{2k+2}, v_{2k+3}, \dots, v_{p-1}, v_p, v_1, x),$$

it follows from connectivity of graph $G(v_{2k+1})$ that $v_{2k+2} \sim v_r$, and, therefore, $\deg v_r = 5$. Now, if $v_{2k} \sim v_{r+1}$, $v_{2k+2} \sim v_{r+1}$or $v_{r-1} \sim v_{r+1}$, then cycle C can be extended to one of the cycles

$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k+2}, v_{2k+3}, \dots, v_{r-1}, v_r, v_{2k}, v_{r+1}, \dots, v_{p-1}, v_p, v_1, x),$$
$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, v_r, v_{r-1}, \dots, v_{2k+3}, v_{2k+2}, v_{r+1}, \dots, v_{p-1}, v_p, v_1, x),$$
$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, v_r, v_{2k+2}, v_{2k+3}, \dots, v_{r-1}, v_{r+1}, \dots, v_{p-1}, v_p, v_1, x),$$

respectively; here P denotes the path $(x, v_3, v_2, v_5, v_4, v_7, v_6, \dots, v_{2k-1})$. This implies that $v_{r+1} \not\sim \{v_{2k}, v_{2k+2}, v_{r-1}\}$ but this contradicts connectivity of graph $G(v_r)$. Thus, the only option for r is the equality $r = 2k+3$. This yields the relation $\{v_{2k}v_{2k+3}, v_{2k+1}v_{2k+3}\} \subset E(G)$ and (i) holds.

Besides, by construction, $\deg v_{2k+1} = 5$ and, since $v_{2k} \not\sim v_{2k+2}$, we have that $G(v_{2k+1}) = P_5 = (v_{2k-2}, v_{2k-1}, v_{2k}, v_{2k+3}, v_{2k+2})$.

Our current goal is to prove the equality $\deg v_{2k} = 3$. Assume that $\deg v_{2k} > 3$. Then, due to connectivity of graph $G(v_{2k})$ there exists a vertex u such that $u \sim \{v_{2k}, v_{2k+3}\}$. In the case that $u \in S$ we derive a contradiction to connectivity of graph $G(v_{2k+3})$. Thus $u \in V(C)$, say $u = v_t$. It is fairly easy to see that $2k+5 \leqslant t \leqslant p-3$; otherwise, we would have a contradiction to the choice of C. Since $v_{2k+4} \not\sim \{v_{2k}, v_{2k+2}\}$, it follows from connectivity of graph $G(v_{2k+3})$ that $v_{2k+4} \sim v_t$. From $\delta(G) \geqslant 3$ and due to connectivity of graph $G(v_{2k+2})$ we derive that $v_{2k+2} \sim v_t$. This results in $t = 2k+5$, since $\Delta(G) = 5$. Thus, we have that $v_{2k+5} \sim \{v_{2k}, v_{2k+2}, v_{2k+3}, v_{2k+4}, v_{2k+6}\}$. Due to connectivity of graph $G(v_{2k+5})$ at least one of the following must be true: $v_{2k} \sim v_{2k+6}$, $v_{2k+2} \sim v_{2k+6}$ or $v_{2k+4} \sim v_{2k+6}$. But then cycle C could be extended to one of the cycles

$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k+2}, v_{2k+3}, v_{2k+4}, v_{2k+5}, v_{2k}, v_{2k+6}, \ldots, v_{p-1}, v_p, v_1, x),$$
$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, v_{2k+3}, v_{2k+4}, v_{2k+5}, v_{2k+2}, v_{2k+6}, \ldots, v_{p-1}, v_p, v_1, x),$$
$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, v_{2k+3}, v_{2k+2}, v_{2k+5}, v_{2k+4}, v_{2k+6}, \ldots, v_{p-1}, v_p, v_1, x),$$

respectively, where $P = (x, v_3, v_2, v_5, v_4, v_7, v_6, \ldots, v_{2k-1})$. Therefore, vertex v_{2k+6} is isolated in graph $G(v_{2k+5})$. This contradiction concludes our consideration of the assumption $\deg v_{2k} > 3$ and, therefore, proves the equality $\deg v_{2k} = 3$. In particular, this means that $G(v_{2k}) = P_3 = (v_{2k-1}, v_{2k+1}, v_{2k+3})$, and (ii) holds. Thus, the proof of Claim 8 is completed. □

We now continue the proof of Theorem 5. Depending on whether p is even or odd we split our further consideration into two cases. In what follows, we keep notation that has been used earlier.

Case 1. p is even.

Looking at the triangle $T_{p-5} = G[v_{p-5}, v_{p-4}, v_{p-3}]$, due to Property 2, we deduce that edge $v_{p-4}v_{p-3}$ belongs to some triangle T_{p-4}, where $T_{p-4} \neq T_{p-5}$. Since by condition of this theorem we have that $\Delta(G) = 5$, it follows from Claims 1 and 8 that either $T_{p-4} = G[v_{p-4}, v_{p-3}, v_{p-2}]$ or $T_{p-4} = G[v_{p-4}, v_{p-3}, v_p]$. This implies that cycle C extends to

$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, \ldots, v_{p-5}, v_{p-6}, v_{p-3}, v_{p-4}, v_{p-2}, v_{p-1}, v_p, v_1, x)$$

or to

$$(x, v_1, v_2, v_3, v_4, \ldots, v_{p-5}, v_{p-4}, v_p, v_{p-3}, v_{p-2}, v_{p-1}, x).$$

which contradicts the choice of C.

Case 2. p is odd.

As in Case 1, we deduce that graph G either contains triangle $G[v_{p-3}, v_{p-2}, v_{p-1}]$ or triangle $G[v_{p-3}, v_{p-2}, v_p]$. This implies that cycle C extends to

$$P \cup (v_{2k-1}, v_{2k-2}, v_{2k+1}, v_{2k}, \ldots, v_{p-4}, v_{p-5}, v_{p-2}, v_{p-3}, v_{p-1}, v_p, v_1, x)$$

or to

$$(x, v_1, v_2, v_3, v_4, \ldots, v_{p-5}, v_{p-4}, v_{p-3}, v_p, v_{p-2}, v_{p-1}, x).$$

which is impossible due to the choice of C .

Thus, graph G does not contain a nonextendable non-Hamilton cycle. Additionally, by Property 1 each vertex of G belongs to some triangle, so that G is fully cycle extendable graph. This completes the proof of the theorem. □

Theorem 5 cannot be improved in the sense that the replacement of the condition $\delta(G) \geqslant 3$ by $\delta(G) \geqslant 2$ does not guarantee even hamiltonicity of graph G. This can be verified by looking either at a complete tripartite graph $K_{1,1,4}$ or at graphs F, G_1, G_2, G_3 shown in Figure 3.

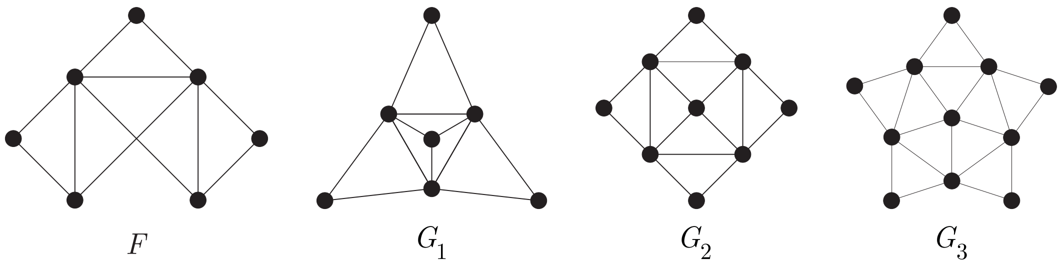

Fig. 3. Graphs F, G_1, G_2 and G_3

Theorem 5 has an interesting corollary. Hendry [15] gave the following conjecture on cycle extendability of chordal graphs (a graph is *chordal* if every cycle of length at least 4 contains a chord).

Conjecture 1 ([15])**.** *All hamiltonian chordal graphs are fully cycle extendable.*

This conjecture is shown to be true for some special classes of chordal graphs, namely split graphs [1], interval graphs [1, 8], planar chordal graphs [16] and some subclasses of strongly chordal graphs [1].

Since each 2-connected chordal graph is locally connected, Theorem 5 implies the following corollary.

Corollary 1. *Conjecture 1 is true for chordal graph G with $\Delta(G) = 5$ and $\delta(G) \geqslant 3$.*

Moreover, Theorem 3 implies that Conjecture 1 is also true for chordal graphs with $\Delta(G) \leqslant 4$.

Theorem 5 can be extended in the following way. Let $\mathcal{F}$ be the class of all connected, locally connected graphs G such that $\Delta(G) = 5$ and G contains four vertices u, v, x and y with the properties $u \sim v$, $\{u, v\} \sim \{x, y\}$, $x \not\sim y$ and $\deg_G x = \deg_G y = 2$. Using similar technique as in the proof of Theorem 5, the following statement can be easily derived (we omit the proof).

Theorem 6. *Let F, G_1, G_2, G_3 be the graphs shown in Figure 3 and G be a connected, locally connected graph such that $\Delta(G) = 5$ and G does not contain graph F as an induced subgraph. Then either G is hamiltonian or $G \in \mathcal{F} \cup \{G_1, G_2, G_3\}$.*

Now we turn to the conditions of hamiltonicity of locally connected graphs under $\Delta(G) > 5$. It is fairly easy to find a connected, locally connected nonhamiltonian graph G with $\Delta(G) = 6$ and $\delta(G) \leqslant 5$. However, an example of a connected, locally connected

6-regular nonhamiltonian graph is more difficult to build. Such a graph appears to have at least 28 vertices [17].

Let $\mathcal{F}(r)$ be the class of connected, locally connected r-regular graphs G such that $r \geqslant 6$ and each edge of G belongs to at least $r-4$ triangles. Kikust [17] has shown that any $G \in \mathcal{F}(r)$ is hamiltonian. Adopting his method, the following stronger statement can be proved.

Theorem 7. *Each graph in class $\mathcal{F}(r)$ is fully cycle extendable.*

4. NP-Completeness

As pointed out in [12], most of graph theory problems can be solved in polynomial time, provided that the maximum vertex degree is fairly small. Indeed, under the assumption that each vertex of a locally connected graph has a degree at most 4, problem HAMILTON CYCLE is trivially solvable in polynomial time due to Theorem 3. Let Δ^* denote such a largest integer that problem HAMILTON CYCLE for an arbitrary locally connected graph G is solvable in polynomial time, provided that $\Delta(G) \leqslant \Delta^*$. It follows from Theorem 3 that $\Delta^* \geqslant 4$. The following theorem implies that $\Delta^* \leqslant 6$.

Theorem 8. *For an arbitrary locally connected graph G with $\Delta(G) \leqslant 7$, problem* HAMILTON CYCLE *is NP-complete.*

Proof. Recall that problem HAMILTON CYCLE is NP-complete for a 2-connected cubic bipartite graph; see [24].

To prove the theorem, consider a connected cubic bipartite graph G. We design a polynomial-time transformation φ of this graph into a locally connected graph $G^* = \varphi(G)$ with $\Delta(G^*) = 7$ such that graph G contains a Hamilton cycle if and only if graph $\varphi(G)$ contains a Hamilton cycle.

Let $G = (X, Y, E)$ be an arbitrary connected cubic graph of order n, where its set of vertices consists of two parts $X = \{x_i \mid 1 \leqslant i \leqslant p\}$, $Y = \{y_j \mid 1 \leqslant j \leqslant p\}$ for $p = n/2$. By a well-known Petersen's theorem (see, e.g., [21]) graph G contains a 1-factor F_1 (i.e., a perfect matching), which can be found in polynomial time. On the other hand, $F_2 = G - F_1$ is a 2-factor of graph G, i.e., its regular spanning subgraph of degree 2. Define an edge 2-coloring $\psi : E(G) \to \{1, 2\}$ by setting

$$\psi(e) = \begin{cases} 1, & \text{if } e \in E(F_1), \\ 2, & \text{if } e \in E(F_2). \end{cases}$$

The transformation φ of graph G into graph $G^* = \varphi(G)$ is as follows:

- Put into correspondence to each vertex $x_i \in X$, $1 \leqslant i \leqslant p$, (correspondingly, $y_j \in Y$, $1 \leqslant j \leqslant p$) of graph G a graph $\varphi(x_i) = K_3$ (correspondingly, graph $\varphi(y_j) = K_3$) with the vertex set $\{x_i^1, x_i^2, x_i^3\}$ (correspondingly, with the vertex set $\{y_j^1, y_j^2, y_j^3\}$);
- Put into correspondence to each edge $x_i y_j$, $1 \leqslant i, j \leqslant p$, of graph G three edges that connect the vertices of the relevant subgraphs $\varphi(x_i)$ and $\varphi(y_j)$, namely, either $x_i^1 y_j^1$, $x_i^2 y_j^2$, and $x_i^2 y_j^1$ (if $\psi(x_i y_j) = 1$) or $x_i^1 y_j^1$, $x_i^2 y_j^2$ and $x_i^1 y_j^2$ (if $\psi(x_i y_j) = 2$).

Thus, graph $G^* = \varphi(G)$ has $3n$ vertices and $15n/2$ edges. It is clear that the described transformation can be implemented in polynomial time. An example of the transformation φ for cubic graph G of order 8 is given in Figure 4, where the edges of 1-factor F_1 are given by the thick lines and all other edges of G belong to 2-factor F_2; the transformation φ is shown only for the marked fragment of G.

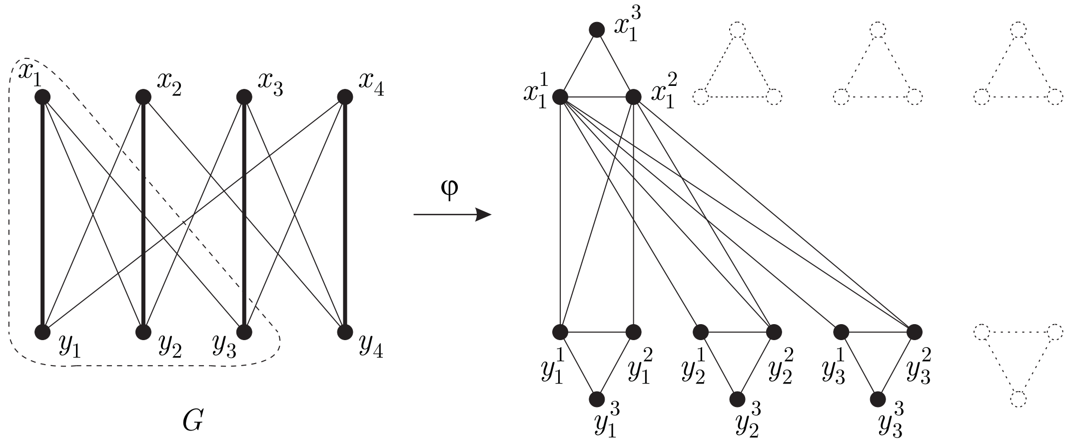

Fig. 4. An illustration to the transformation φ

We now show that G^* is a locally connected graph. Consider an arbitrary vertex $x \in \{x_i^1, x_i^2, x_i^3\}$, $1 \leqslant i \leqslant p$. If $x = x_i^3$, then obviously $G^*(x)$ is isomorphic to K_2 and is connected. Let $x = x_i^2$ and $N_G(x_i) = \{y_j, y_k, y_l\}$. Without loss of generality, we may assume that $\psi(x_i y_j) = 1$. Then we have that $N_{G^*}(x) = \{x_i^1, x_i^3, y_j^1, y_j^2, y_k^2, y_l^2\}$, and since by construction $\psi(x_i y_k) = \psi(x_i y_l) = 2$, it follows that $x_i^1 \sim \{x_i^3, y_j^1, y_k^2, y_l^2\}$ and $y_j^1 \sim y_j^2$. Thus, graph $G^*(x)$ is isomorphic to graph L_1, which is obtained from a star $K_{1,4}$ by subdividing one of its edges once. Now, let $x = x_i^1$ and let, as above, $N_G(x_i) = \{y_j, y_k, y_l\}$, where without loss of generality, $\psi(x_i y_k) = \psi(x_i y_l) = 2$. Then $N_{G^*}(x) = \{x_i^2, x_i^3, y_j^1, y_k^1, y_k^2, y_l^1, y_l^2\}$, and since $\psi(x_i y_j) = 1$, it follows that $x_i^2 \sim \{x_i^3, y_j^1, y_k^2, y_l^2\}$ and $y_k^1 \sim y_k^2$, $y_l^1 \sim y_l^2$. Thus, graph $G^*(x)$ is isomorphic to graph L_2, which is obtained from a star $K_{1,4}$ by subdividing two of its edges once each. The considered cases exhaust all possible situations, no matter whether $x \in \{x_i^1, x_i^2, x_i^3\}$ or $x \in \{y_j^1, y_j^2, y_j^3\}$. We therefore conclude that graph G^* is locally connected. Besides, taking into account that $\boldsymbol{N}(G^*) = \{K_2, L_1, L_2\}$ we derive that $\Delta(G^*) = 7$.

It is left to demonstrate that graph G is hamiltonian if and only if graph $G^* = \varphi(G)$ is hamiltonian.

Let C be a Hamilton cycle in graph G. Since G is bipartite, we may assume that $C = (x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_p}, y_{j_p}, x_{i_1})$, where the vertices are listed in the order of traversing this cycle. But then cycle C corresponds to the following Hamilton cycle

$$(x_{i_1}^2, x_{i_1}^3, x_{i_1}^1, y_{j_1}^1, y_{j_1}^3, y_{j_1}^2, x_{i_2}^2, x_{i_2}^3, x_{i_2}^1, y_{j_2}^1, y_{j_2}^3, y_{j_2}^2, \ldots, x_{i_p}^2, x_{i_p}^3, x_{i_p}^1, y_{j_p}^1, y_{j_p}^3, y_{j_p}^2, x_{i_1}^2)$$

in graph $\varphi(G)$.

Suppose now that C^* is a Hamilton cycle in graph $G^* = \varphi(G)$. Since $\deg_{G^*} x_i^3 = \deg_{G^*} y_j^3 = 2$, we deduce that the paths $P_i^x = (x_i^1, x_i^3, x_i^2)$ and $P_j^y = (y_j^1, y_j^3, y_j^2)$, $1 \leqslant i, j \leqslant p$, must belong to cycle C^*. Without loss of generality, assume that $P_{i^*}^x$ and $P_{j^*}^y$, $1 \leqslant i^*, j^* \leqslant p$, are the "neighboring" paths in C^*. Traversing C^*, we enter $\varphi(x_{i^*})$ via some edge and traverse all vertices of $\varphi(x_{i^*})$ along the path $P_{i^*}^x$ (in one of the two possible directions), then leave $\varphi(x_{i^*})$ via one of the three edges that connect $\varphi(x_{i^*})$ and $\varphi(y_{j^*})$ (this edge corresponds to the edge $x_{i^*}y_{j^*}$ in graph G). Having thereby entered $\varphi(y_{j^*})$, traverse that graph along the path $P_{j^*}^y$ (in one of the two possible directions) and then leave $\varphi(y_{j^*})$ via some edge. Thus, contracting the edges of each subgraph $\varphi(x_i)$ and $\varphi(y_j)$, $1 \leqslant i, j \leqslant p$, we transform a Hamilton cycle in graph $\varphi(G)$ into a Hamilton cycle in graph G. This proves the theorem. □

Thus, Theorems 3 and 8 imply that the inequalities $\Delta^* \geqslant 4$ and $\Delta^* \leqslant 6$ respectively hold. Combining these two conditions we obtain the lower and upper bounds on Δ^*, given by $4 \leqslant \Delta^* \leqslant 6$. We suppose that the following conjecture is true.

Conjecture 2. $\Delta^* = 6$.

We conclude this section by pointing out that Theorem 8 holds even for 7-regular locally connected graphs. The proof of this results follows the same lines as the proof of Theorem 8. However, in the description of the procedure for constructing graph $G^* = \varphi(G)$, a triangle $\varphi(x_i)$, $1 \leqslant i \leqslant p$, (correspondingly, a triangle $\varphi(y_j)$, $1 \leqslant j \leqslant p$) is extended to a locally connected graph H_i^x (correspondingly, H_j^y), in which all vertices have degree 7, except x_i^1 and x_i^2 (correspondingly, y_j^1 and y_j^2). Additionally, $\deg_{H_i^x} x_i^1 = \deg_{H_j^y} y_j^1 = 2$, (correspondingly, $\deg_{H_i^x} x_i^2 = \deg_{H_j^y} y_j^2 = 3$) and H_i^x (correspondingly, H_j^y) contains a Hamilton (x_i^1, x_i^2)-path (correspondingly, a Hamilton (y_j^1, y_j^2)-path).

5. **Conclusion**

In this paper, we consider hamiltonian properties (the existence of Hamilton cycles and fully cycle extendability) of locally connected graphs with bounded vertex degree. We explicitly describe all connected, not necessary finite, locally connected graphs with $\Delta(G) \leqslant 4$ (Theorem 3) thereby generalizing the results by Chartrand, Pippert [6] and (in case of $\Delta(G) \leqslant 4$) by Hendry [14].

We have shown that every connected, locally connected graph with $\Delta(G) = 5$ and $\delta(G) \geqslant 3$ is fully cycle extendable (Theorem 5) which extends the results by Kikust [18] and Hendry [14]. Moreover, we show that Hendry's conjecture [15] on cycle extendability of hamiltonian chordal graphs is true for any chordal graphs with $\Delta(G) = 5$, $\delta(G) \geqslant 3$ and for chordal graphs with $\Delta(G) \leqslant 4$. In addition, we describe new classes of locally connected graphs with bounded vertex degree which are hamiltonian (Theorems 6 and 7).

At last, we prove that the HAMILTON CYCLE problem for locally connected graphs with $\Delta(G) \leqslant 7$ is NP-complete (Theorem 8) and show that this result holds even for 7-regular locally connected graphs.

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