

ON LIAPUNOV TRANSFORMATIONS OF LINEAR
SYSTEMS OF IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. An asymptotical equivalence of linear systems of implicit differential equations is investigated. Sufficient conditions to transform a linear system of implicit differential equations to a system of special type are given

We shall consider a system of implicit linear differential equations

$$(1) \quad A(t)\dot{x} + B(t)x = 0, \quad t \in R_+ = [0, +\infty]$$

where $A(t)$, $B(t)$ are $n \times n$ matrix functions of a real variable t and x is an n -dimensional vector-function. We shall call an $n \times n$ matrix $X(t)$ a fundamental solution matrix of the system (1) if for any constant k -dimensional vector c the function

$$(2) \quad x(t) = X(t)c$$

is a solution of (1) and any solution $x(t)$ of (1) has the form (2) with some constant vector c . We shall also call the system (1) asymptotically equivalent to the system

$$(3) \quad P(t)\dot{y} + Q(t)y = 0, \quad t \in R_+,$$

if there is an $n \times n$ Liapunov matrix $L(t)$ (see [1]) such that for any solution $y(t)$ of (3) the function $x(t) = L(t)y(t)$ is a solution of (1) and for any solution $x(t)$ of (1) the function $y(t) = L^{-1}(t)x(t)$ is a solution of (3). It follows (see [5]) that the system (1) is asymptotically equivalent to the system (3) if and only if there is a fundamental solution matrix $X(t)$ of (1) such that $X(t) = L(t)Y(t)$ where $L(t)$ is a Liapunov matrix and $Y(t)$ is a fundamental solution matrix of (3).

We shall consider the systems (1) and (3) when the matrices $A(t)$ and $P(t)$ are singular on R_+ , i.e., $\det A(t) = \det P(t) = 0$ for all $t \geq 0$.

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Theorem. Let A and B be absolutely continuous and bounded together with their first derivatives almost everywhere on R_+ . Let also there exist a $k \times k$ submatrix $A_0(t)$ of the matrix $A(t)$ such that

$$(4) \quad \inf_{t \geq 0} |\det A_0(t)| > 0,$$

$$(5) \quad \text{rank } A(t) = k \quad \text{for all } t \geq 0,$$

$$(6) \quad \inf_{t \geq 0} \left| \frac{\partial^k}{\partial \lambda^k} \det(A(t)\lambda + B(t)) \right| > 0.$$

Then the system (1) is asymptotically equivalent to the system (3) where the matrices $P(t)$ and $Q(t)$ have the forms

$$(7) \quad P = \begin{pmatrix} O_1 & O_2 \\ O_3 & E_0 \end{pmatrix}, \quad Q = \begin{pmatrix} E_1 & O_2 \\ O_3 & Q_0 \end{pmatrix},$$

O_1, O_2, O_3 are zero $(n-k) \times (n-k)$, $(n-k) \times k$, $k \times (n-k)$ matrices, E_1, E_0 are identity $(n-k) \times (n-k)$, $k \times k$ matrices, respectively, and Q_0 is a $k \times k$ matrix summable and bounded a.e. on R_+ .

Proof. Without losing generality we may assume that the submatrix A_0 is in the right-down corner of the matrix A . Then (4), (5) imply that the matrix A can be written in the form

$$A = \begin{pmatrix} C_1 A_0 C_3 & C_1 A_0 \\ A_0 C_3 & A_0 \end{pmatrix}$$

where C_1 and C_3 are respectively $(n-k) \times k$ and $k \times (n-k)$ matrices absolutely continuous and bounded together with their first derivatives a.e. on R_+ .

Denote

$$T = \begin{pmatrix} E_1 & -C_1 \\ O_3 & A_0^{-1} \end{pmatrix}, \quad S = \begin{pmatrix} E_1 & O_2 \\ -C_3 & E_0 \end{pmatrix}.$$

Multiplying (1) on the left by the nonsingular matrix T and making the change of variables $x = Sz$ we obtain the system

$$(8) \quad P(t)\dot{z} + F(t)z = 0$$

where P is defined by (7) and

$$(9) \quad F = T(AS + BS).$$

It is easy to check that S is a Liapunov matrix and thus (8) is asymptotically equivalent to (1).

We can consider the system (8) as two systems

$$(10) \quad F_1(t)z_1 + F_2(t)z_2 = 0$$

$$(11) \quad \dot{z}_2 + F_3(t)z_1 + F_0(t)z_2 = 0$$

where

$$(12) \quad F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_0 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

F_1, F_2, F_3, F_0 are $(n-k) \times (n-k)$, $(n-k) \times k$, $k \times (n-k)$, $k \times k$ matrices, z_1, z_2 are $(n-k)$ - and k -dimensional vectors respectively.

It is seen from (9) that

$$(13) \quad F = G + TBS$$

where

$$(14) \quad G = PS^{-1}\dot{S} = \begin{pmatrix} O_1 & O_2 \\ -C_3 & O_0 \end{pmatrix},$$

O_0 is zero $k \times k$ matrix.

From (9), (12), (13) we obtain

$$(15) \quad \det(A\lambda + B) = \det T^{-1} \det(P\lambda + F - G) \det S^{-1} = \det A_0 \det(P\lambda + F - G).$$

The relations (7), (12), (14) imply

$$(16) \quad \det(P\lambda + F - G) = \lambda^k \det F_1 + \sum_{i=0}^{k-1} f_i \lambda^i$$

where $f_i, i = 1, \dots, k-1$, are functions of real variable t .

Hence taking (5), (15), (16) into account we have

$$\inf_{t \geq 0} |\det F_1(t)| > 0$$

and thus from (10) and (11) we get

$$(17) \quad z_1 = -F_1^{-1}F_2z_2,$$

$$(18) \quad \dot{z}_2 = (F_3F_1^{-1}F_2 - F_0)z_2.$$

Let z_2 be a fundamental solution matrix of the system (18). Then

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

is a fundamental solution matrix of the system (8) where, in view of (17),

$$(19) \quad Z_1 = -F_1^{-1} F_2 Z_2.$$

Consider the system (3) where P, Q are given by (7) with $Q_0 = F_0 - F_3 F_1^{-1} F_2$ that is summable and bounded a.e. on R_+ . If Y is a fundamental solution of (3) then there exists a constant $k \times k$ matrix C_0 such that (see (19))

$$Y = LZC_0 = \begin{pmatrix} E_1 & F_1^{-1} F_2 \\ O_3 & E_0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} C_0.$$

From (9), (13), (14) we have that F_1, F_2 are absolutely continuous and bounded together with their first derivatives a.e. on R_+ . Thus L is a Liapunov matrix. Consequently, (3) is asymptotically equivalent to (8) and hence (3) is asymptotically equivalent to (1). The theorem is proved. \square

Corollary. *If A and B satisfy the conditions of Theorem then the system (1) is asymptotically equivalent to a system with piecewise constant coefficients. Moreover, the sequence (t_n) , where t_n are points of discontinuity of coefficients, is a subsequence of the sequence (ml) , where l is a positive number, $m = 1, 2, 3, \dots$*

Proof. By Theorem the system (1) is asymptotically equivalent to the system (3) where P and Q are of the form (7). Let

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where y_1, z_1 are $(n - k)$ -dimensional, y_2, z_2 are k -dimensional vectors. Then (see [3]) the system

$$(20) \quad \dot{y}_2 + Q_0 y_2 = 0$$

is asymptotically equivalent to the system

$$(21) \quad \dot{z}_2 + R_0 z_2 = 0$$

where R_0 is a piecewise constant $k \times k$ matrix whose points of discontinuity make a subsequence of the sequence (ml) . Thus

$$(22) \quad Y_2 = L_0 Z_2$$

where Y_2, Z_2 are fundamental solution matrices of (20) and (21) respectively and L_0 is a Liapunov matrix. From (22) we have $Y = LZ$, where

$$Y = \begin{pmatrix} O_1 \\ Y_2 \end{pmatrix}, \quad Z = \begin{pmatrix} O_1 \\ Z_2 \end{pmatrix}$$

are fundamental solution matrices of (3) and the system

$$(23) \quad \begin{pmatrix} O_1 & O_2 \\ O_3 & E_0 \end{pmatrix} \dot{z} + \begin{pmatrix} E_1 & O_2 \\ O_3 & R_0 \end{pmatrix} z = 0$$

respectively, and

$$L = \begin{pmatrix} E_1 & O_2 \\ O_3 & L_0 \end{pmatrix}$$

is a Liapunov matrix. Thus the system (3), and hence the system (1), is asymptotically equivalent to the system (23). Corollary is proved. \square

For other results connected with transformations of (1) to systems of special types see [4,5].

We close this article with some examples which show that the conditions of Theorem are only sufficient but not necessary.

Example 1. Consider the system (1) in which

$$(24) \quad A = \begin{pmatrix} r \cos \varphi & \varrho \cos \psi \\ r \sin \varphi & \varrho \sin \psi \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where the functions $r, \varrho, \varphi, \psi, b_{ij}, i, j = 1, 2$, are absolutely continuous and bounded together with their first derivatives a.e. on R_+ . The condition $\det A(t) = 0$ for all $t \geq 0$ implies

$$(25) \quad r(t)\varrho(t) \sin(\varphi(t) - \psi(t)) = 0 \text{ for all } t \geq 0.$$

Let $\text{rank } A(t) = k = 1$ for all $t \geq 0$, then $R^2(t) = r^2(t) + \varrho^2(t) \neq 0$ for all $t \geq 0$. Assume that

$$(26) \quad \inf_{t \geq 0} R^2(t) > 0$$

but there is no 1×1 submatrix A_0 for which the condition (4) is fulfilled. Let also the condition (5) and (6) be valid.

Denote $T_r = \{t \in R_+ | r(t) = 0\}$ and $T_\varrho = \{t \in R_+ | \varrho(t) = 0\}$. The relations (25), (26) imply $T_r \cap T_\varrho = \emptyset$ and $\varphi(t) = \psi(t) + k(t)\pi$ for all $t \in R_+ \setminus (T_r \cup T_\varrho)$ where $k(t)$ is a piecewise constant function which admits only integer values. It is

not difficult to verify that there are piecewise constant functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ admitting only integer values such that the function

$$\omega(t) = \begin{cases} \varphi(t) + k_1(t)\pi, & t \in T_\varrho, \\ \psi(t) + k_2(t)\pi, & t \in T_r, \\ \varphi(t) + k_3(t)\pi, & t \in R_+ \setminus (T_r \cup T_\varrho) \end{cases}$$

is absolutely continuous and bounded together with its first derivatives a.e. on R_+ . Define

$$\alpha(t) = \begin{cases} k_1(t), & t \in T_\varrho, \\ k_3(t), & t \in T_r, \\ a(t), & t \in R_+ \setminus (T_r \cup T_\varrho), \end{cases} \quad \beta(t) = \begin{cases} b(t), & t \in T_\varrho, \\ k_2(t), & t \in T_r, \\ k_3(t) + k(t), & t \in R_+ \setminus (T_r \cup T_\varrho), \end{cases}$$

where $a(t)$, $b(t)$ are arbitrary functions such that $\alpha(t)$, $\beta(t)$ are continuous on R_+ . Denote

$$T = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix}, \quad S = \begin{pmatrix} (-1)^{\alpha r} & (-1)^{\beta-1} \varrho \\ (-1)^{\beta} \varrho & (-1)^{\alpha r} \end{pmatrix}.$$

The matrices T and S are absolutely continuous and bounded together with their first derivatives a.e. on R_+ . Moreover, $\det T(t) = 1$ and $\det S(t) = r^2(t) + \varrho^2(t) = R^2(t)$ for all $t \in R_+$. From (26) we see that S is a Liapunov matrix and therefore the system (1) with the matrices (24) is asymptotically equivalent to the system

$$(27) \quad T A S \dot{y} + (T A \dot{S} + T B S) y = 0$$

where

$$T A S = \begin{pmatrix} R^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $T B S = (c_{ij})$, $i, j = 1, 2$. Then $\det(T A S \lambda + T A \dot{S} + T B S) = R^2 c_{22} \lambda + f$ where f is a function of the variable t . On the other hand,

$$R^2 \det(A \lambda + B) = \det T \det(A \lambda + B) \det S = \det(T A S \lambda + T B S) = R^2 c_{22} \lambda + g$$

where g is a function of the variable t . Thus the conditions (6) and (26) imply

$$\inf_{t \geq 0} \left| \frac{\partial}{\partial \lambda} \det(T A S \lambda + T A \dot{S} + T B S) \right| > 0.$$

Therefore, the matrices $T A S$ and $(T A \dot{S} + T B S)$ satisfy the conditions of Theorem. Thus the system (27), and hence the system (1) with the matrices (24), is asymptotically equivalent to a system of the type (3) with P and Q of the form (7).

Example 2. Consider (1) with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\det A_0(t) = 1$ and $\text{rank } A(t) = 2$ for all $t \geq 0$, i.e., the conditions (4) and (5) are fulfilled. But $\det(A\lambda + B) = \lambda + 1$ and hence the condition (6) is not valid. However, since in this case A and B are constant matrices and $\deg \det(A\lambda + B) = 1$ then (see [2, Kapitel 12]) the system (1) is asymptotically equivalent to the system $\text{diag } \{0, 0, 1\} \dot{z} + \text{diag } \{1, 1, 1\} z = 0$ which has the type (3) with (7).

Example 3. Now we show that the condition $\deg \det(A\lambda + B) = \text{const}$ (see [2, Kapitel 12]) is not necessary, too. Consider (1) where now

$$A = \begin{pmatrix} \sin t - \cos t & \sin t & 0 \\ \sin t & \sin t - \cos t & \cos t \\ 1 + \cos t - \sin t & -1 - \sin t & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} (\sin t + \cos t)(1 + \cos t) & \cos t(1 + \cos t) & 0 \\ \sin t + 2 \cos^2 t & \cos t(2 + \cos t - \sin t) & -\cos t \\ -1 - (\sin t + \cos t)(1 + \cos t) & 1 - \cos t(1 + \cos t) & -1 \end{pmatrix},$$

Then $\det(A(t)\lambda + B(t)) = (1 - \lambda) \cos t(1 + \cos t)$, however the system (1) in this case has the fundamental solution matrix $X(t) = (0, 0, e^t)^T$ and hence (1) is obviously asymptotically equivalent to the system $\text{diag } \{0, 0, 1\} \dot{z} - \text{diag } \{1, 1, 1\} z = 0$ which has the type (3) with (7).

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