Математика и информатика

УДК 517.925

B.S. KALITINE (FRANCE), J.-C. VIVALDA (BELARUS)

A THEOREM OF UNSTABILITY*

Исследуется проблема неустойчивости состояний равновесия автономных дифференциальных систем и автономных дискретных систем. Приведено обобщение теоремы Н.Н. Красовского со знакопостоянной производной функции Ляпунова по времени в случае, когда множество, где эта производная обращается в нуль, может содержать положительные полутраектории. Доказана соответствующая теорема обращения для линейных систем дифференциальных уравнений.

1. Introduction. The well known theorem of the direct method of A.M. Lyapunov [1] gives a criterion for the instability of the equilibrium of a differential system

$$\dot{x} = f(x), \tag{1}$$

where f is a mapping defined on an open set $G \subset \mathfrak{R}^n$ with $0 \in G$ and $f(0)=0$. This criterion is based on the existence of a positive definite function V defined on G such that its derivative $\dot{V}$ along the trajectories of system (1) is positive definite. A generalization of this result was obtained by H.G. Tchetaev [2] and N.N. Krassovskii [3] and others. As far as we are concerned with the result in [3], the hypothesis are weakened in the following way: the derivative of V is only assumed to be positive but it is also assumed that the set $E=\{x \in G: \dot{V}(x) = 0\}$ does not contain positive trajectories except for the origin.

The purpose of this paper is to give a generalization of this result of Krassovskii [3] about the instability of differential systems for continuous time systems as well as for discrete time ones.

A point x_0 being given, we denote by $x(t, x_0)$ the solution of (1) issued from x_0; we denote also by $\gamma^+(x_0)$ (resp. $\gamma^-(x_0)$) the positive (resp. the negative) semi-trajectory issued from x_0: $\gamma^+(x_0)=\{x(t, x_0): t \geq 0\}$. Also we denote by $B(x_0, r)$ the ball of center x_0 and radius r.

2. Main results. First, we give our generalized version of the above-mentioned result of Krassovskii.

Theorem 1. *Consider the differential equation* $\dot{x} = f(x)$ *defined in a neighborhood U of the origin in* $\mathfrak{R}^n$; *we assume that f is lipschitzian and satisfies* $f(0)=0$. *Moreover, we assume that there exists a function V of class* C^1 *defined in* $U \setminus \{0\}$ *such that*:

i) $V(0)=0$;

ii) *for every* $\alpha>0$, *there exists* $p \in B(0, \alpha)$ *such that* $V(p)>0$;

iii) *for every* $x \in U$, $\dot{V}(x) \geq 0$;

iv) *for every* $x_0 \in U$, $\gamma^+(x_0) \not\subset E=\{x \in U: \dot{V}(x) = 0$ *and* $V(x) \neq 0\}$.

Then $x=0$ *is an unstable equilibrium point for* $\dot{x} = f(x)$.

Note first that the conditions of application of this theorem are identical to the ones of Krassovskii [3] except for what concerns the fourth point where our set E is larger than the one appearing in the result of Krassovskii.

Proof. Take $\varepsilon_0>0$ such that $B(0, \varepsilon_0) \subset U$, let $0<\alpha<\varepsilon_0$ and take $x_0 \in B(0, \alpha)$ such that $V(x_0)>0$. Assume that $x(t, x_0)$ remains in $B(0, \varepsilon_0)$ for all $t \geq 0$, then the ω-limit set of x_0 is nonempty; denoting this set by $L^+(x_0)$, we

---

* Статья публикуется в авторской редакции.

can say that V is constant and $\dot{V} \equiv 0$ on $L^+(x_0)$. To be more precise, since $V(x(t, x_0)) \geq V(x_0) > 0$ this constant is positive. Hence we can say that $L^+(x_0) \subset E$; now, if we take a point $y \in L^+(x_0)$, we have $\gamma^+(y) \subset L^+(x_0) \subset E$ which is a contradiction.

Remark that, since an ω-limit set contains all the semi-trajectories issued from one of its points, point *iv*) of the theorem could be replaced by

iv′) for every $x_0 \in U$, $\gamma^-(x_0) \not\subset E = \{x \in U$: $\dot{V}(x) = 0$ and $V(x) \neq 0\}$.

We state now a theorem of instability concerned with the discrete time systems.

Let U be an open neighborhood of the origin in $\Re^n$ and f: $U \to \Re^n$ a continuous function such that $f(0)=0$. We consider the discrete time system

$$x_{k+1} = f(x_k), \tag{2}$$

where $x_k \in U$ and $k \in \boldsymbol{N}$. We denote by $x(k, x_0)$ ($k=0, 1, 2, \ldots$) the solution of system (2) with initial condition x_0, so, we have $x(0, x_0)=x_0$ and $x(k+1, x_0)=f(x(k, x_0))$. It is well known [11] that if the solution $x(x_0, k)$ is bounded on $\boldsymbol{N}$, its ω-limit set is nonempty, compact and invariant. In the definition below, we introduce the notion of stability for discrete time systems.

Definition. The fixed point $x_0=0$ is said to be stable if

$$(\forall \varepsilon>0)(\exists \delta>0)(\forall x_0 \in B(0, \delta))(\forall k \in \boldsymbol{N}): \|x(k, x_0)\| < \varepsilon;$$

it is said unstable if it is not stable.

Concerning discrete time systems, the second method of Lyapunov was used for the first time in [6, 7]; subsequent developments can be found in [9–11]. In particular, in [8], the author states a result, analogous to the theorem of Barbachin – Krassovskii on the global asymptotic stability, for the solutions of a discrete time system by using a Lyapunov function which is non increasing along the solutions of the system. In [9–11], the authors use the method of positive semi-definite Lyapunov functions for discrete time systems. In these works, the authors study the qualitative theory of the stability of the solutions and the related trajectories. In particular, they introduce the notion of invariant sets and they show that if a solution of system (2) is bounded, its ω-limit set is nonempty, compact and invariant. They also show that, if there exists a trajectory included in a compact set $K \subset U$ and a function V: $K \to \Re^+$ whose is non increasing along this trajectory, then all the limit points of this trajectory are located an the same level surface of function V. Using these results, we can show the following result, analogous to Theorem 1.

Theorem 2. *Assume that there exist a neighborhood U of the origin and a continuous function V: $U \to \Re$, such that the following conditions are satisfied*:

i) $V(0)=0$;

ii) *given* $\alpha>0$ *there exists* $p \in B(0, \alpha)$, *such that* $V(p)>0$;

iii) $V(x(k+1, x_0)) \geq V(x(k, x_0)) \geq 0$ *for every solution* $x(k, x_0) \in U$ *and for every* $k=1, 2, \ldots$;

iv′) *for every* $x_0 \in U$, *we have* $\gamma^+(x_0) \not\subset E = \{x \in U$: $V(x) = V(f(x))$ *and* $V(x) \neq 0\}$.

Then 0 *is an unstable equilibrium point of system* (2).

Proof. The proof is analogous to the one of Theorem 1. Take $\varepsilon_0>0$ such that $B(0, \varepsilon_0) \subset U$, let $0<\alpha<\varepsilon_0$ and take $x_0 \in B(0, \alpha)$ such that $V(x_0)>0$. Assume that $x(k, x_0)$ remains in $B(0, \varepsilon_0)$ for every $k=1, 2, 3, \ldots$, then, $L^+(x_0)$, the ω-limit set of x_0 is nonempty.

Take $x^* \in L^+(x_0)$, x^* is the limit of a subsequence of the sequence $(x(k, x_0))_{k \geq 0}$: $x^* = \lim_{i \to \infty} x(k_i x_0)$ where (k_i) is a increasing sequence of integers tending to infinity. Function V being nondecreasing along the solutions of (2), we have $V(f(x^*)) \geq V(x^*)$ and $V(x^*) \geq V(x(k_i, x_0))$ for every index i (notice that this inequality implies that $V(x^*) \geq V(x_0) > 0$). Due to the continuity of f, we have

$$f(x^*) = \lim_{i \to \infty} x(k_i + 1, x_0). \tag{3}$$

Now for every index i, there exists an index j such that $k_j \geq k_i + 1$ and so

$$V(x(k_i + 1, x_0)) \leq V(x(k_i, x_0)) \leq V(x^*). \tag{4}$$

Function V being continuous, from (3) we deduce that

$$V(f(x^*)) = \lim_{i \to \infty} V(x(k_i + 1, x_0)).$$

This equality combined with inequality (4) gives

$$V(f(x^*)) \leq V(x^*).$$

So we can conclude that V is positive on $L^+(x_0)$ and that $V=V\circ f$ on this set, which means that $L^+(x_0)\subset E$. But if we take $x^*\in L^+(x_0)$, we have $\gamma^+(x^*)\subset L^+(x_0)\subset E$ which is a contradiction which allows us to conclude that there exists indices k such that $x(k, x_0)\notin B(0, \varepsilon_0)$.

3. Converse of the instability result in the linear case. The converse of the theorem of Krassovskii can be found in [5, 12], we consider here the case of a linear equation

$$\dot{x} = Ax \tag{5}$$

and we state the following theorem.

Theorem 3. *If the origin is an unstable equilibrium in equation* (5), *there exists a quadratic form* $x \to Q(x)$ *which satisfies all the assumption of Theorem* 1.

Proof. If the equilibrium of system (5) is unstable then

1) one of the eigenvalues of A has a positive real part;

2) or there exists a nonzero nilpotent block in the Jordan decomposition of A;

3) or A has an eigenvalue λ which is a pure imaginary number and whose algebraic multiplicity is not equal to its geometric multiplicity.

In the first case, assume that λ is an eigenvalue of A with positive real part; notice that λ is also an eigenvalue of A' the transposed matrix of A. If $\lambda\in\Re$, let v be an eigenvector of A' related to λ and consider the quadratic form defined by $Q(x)=(x'v)^2$. We have

$$\dot{Q}(x) = 2(x'v)((Ax)'v) = 2(x'v)(x'(A'v)) = 2\lambda(x'v)^2$$

so $\dot{Q}$ is obviously nonnegative and the set E defined in Theorem 1 is empty.

If $\lambda\notin\Re$, we can write $\lambda=a+ib$ with $a>0$. There exist two nonzero vectors v_1 and v_2 such that

$$A'v_1 = av_1 + bv_2,\quad A'v_2 = av_1 - bv_2$$

we define Q by $Q(x)=(x'v_1)^2 + (x'v_2)^2$ and we have

$$\dot{Q}(x) = 2(x'v_1)((Ax)'v_1) + 2(x'v_2)((Ax)'v_2) = 2a((x'v_1)^2 + (x'v_2)^2).$$

So, we can see that $\dot{Q}$ is nonnegative and that the set E is empty.

In the second case, we can find nonzero vectors v_1 and v_2 such that

$$A'v_1 = 0,\quad A'v_2 = v_1,$$

we define Q by $Q(x)=(x'v_1)(x'v_2)$ and we have

$$\dot{Q}(x) = ((Ax)'v_1)(x'v_2) + (x'v_1)((Ax)'v_2) = (x'v_1)^2$$

and so $\dot{Q}$ is nonnegative and the set E is empty. In the last case, let $\lambda=ia$ with $a\neq 0$; there exist nonzero vectors $v_1,\ldots,v_4$ such that

$$A'v_1 = av_2,\quad A'v_2 = -av_1,\quad A'v_3 = v_1 + av_4,\quad A'v_4 = v_2 - av_3,$$

we define Q by $Q(x)=(x'v_1)(x'v_3)+(x'v_2)(x'v_4)$ and we have

$\dot{Q}(x) = ((Ax)'v_1)(x'v_3) + (x'v_1)((Ax)'v_3) + ((Ax)'v_2)(x'v_4) + (x'v_2)((Ax)'v_4) = (x'v_1)^2 + (x'v_2)^2$ and so $\dot{Q}$ is nonnegative and E is empty.

4. Examples.

1.1. Continuous system. Consider the Liénard's equation

$$\ddot{x} + (x^4 - 2\dot{x}^2)^{2p}\dot{x} - x^3 = 0, \tag{6}$$

where $p\in\boldsymbol{N}^*$ and it is assumed that $a(0)=b(0)=0$. Together with this equation, we consider the function V defined by

$$V(x,\dot{x}) = -\frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4,$$

we have

$$\dot{V}(x,\dot{x}) = \dot{x}^2(x^4 - 2\dot{x}^2)^{2p}$$

so $\dot{V}$ is nonnegative and

$$E = \{(x,\dot{x}):\ \dot{V}(x,\dot{x}) = 0\} = \{(x,\dot{x}):\ \dot{x} = 0\} \cup \{(x,\dot{x}):\ x^4 - 2\dot{x}^2 = 0\}.$$

Let $(x(t)\dot{x}(t))$ be a solution of (6) included in E, then we have $\ddot{x} = x^3$, from which we deduce

$$\frac{d}{dt}(x^4 - 2\dot{x}^2) = 4\dot{x}x^3 - 4\dot{x}\ddot{x} = 4\dot{x}\ddot{x} - 4\dot{x}x^3 = 0$$

so the expression $x^4(t)-2\dot{x}^2(t)$ is constant, if this constant is nonzero, the inclusion $(x(t)\dot{x}(t))\in E$ for all $t\geq 0$ implies that $\dot{x}(t)=0$ for all t, which implies that $\ddot{x}(t)=x^3(t)=0$ for all t, so $(x(t), y(t))$ is the null solution of (6). If this constant is zero, we have $x^4(t)-2\dot{x}^2(t)=0$ for all t, then $V(x(t)\dot{x}(t))=0$ for all t. In conclusion, we proved that there does not exist solutions of (6) included in the set $E\cap\{(x,\dot{x}):\ \dot{V}(x,\dot{x})>0\}$. Nevertheless, notice that $(x(t), y(t))$ defined by

$$x(t)=-\frac{\sqrt{2}}{1+t},\quad y(t)=\frac{\sqrt{2}}{(1+t)^2}$$

is a solution of (6) included in $E\cap\{(x,\dot{x}):\ \dot{V}(x,\dot{x})>0\}$ and so, we cannot conclude here thanks to the above-mentioned theorem of Krassovskii.

4.2. Discrete time system. Consider the system given in $\Re^2$

$$\begin{cases} x_{k+1}=y_k+2\alpha x_k y_k, \\ y_{k+1}=x_k+x_k^2+\alpha^2 y_k^2 \end{cases} \tag{7}$$

function f defining this system is given by

$$f(x,y)=\begin{pmatrix} y+2\alpha xy \\ x+x^2+\alpha^2 y^2 \end{pmatrix}.$$

Moreover, if $\alpha=0$, U is chosen to be equal to $\Re^n$, while if $\alpha\neq 0$, we take for U an open neighborhood of the origin which does not contain the point of coordinates $(2\alpha^2(1+\alpha), -2\alpha(1+\alpha))$. Taking the Liapunov function $V(x, y)=x+y$, we have

$$\Delta V(x,y)=V(f(x,y))-V(x,y)=(x+\alpha y)^2\geq 0.$$

Set E is

$$E=\{(x,y)\in U:\ x=-\alpha y \text{ and } x\neq -y\},$$

which is obviously empty if $\alpha=1$. If $\alpha\neq 1$, if a solution (x_k, y_k) of (7) remains on E, we have $x_k=-\alpha y_k$ for every index $k=0, 1, 2, \ldots$ and the sequence $(y_k)_{k\geq 0}$ satisfies the relations

$$-\alpha y_{k+1}=y_k-2\alpha^2 y_k^2,\quad y_{k+1}=-\alpha y_k+2\alpha^2 y_k^2,$$

from which we deduce that

$$0=(1-\alpha^2)y_k+2\alpha^2(1-\alpha)y_k^2$$

and so $(x_k, y_k)=(0,0)$ or

$$y_k=\frac{1+\alpha}{2\alpha^2},\quad x_k=-\frac{1+\alpha}{2\alpha}$$

for every index $k=0, 1, \ldots$, but the second possibility cannot occur because of the choice of U, so set E does not contain any positive semi-trajectory of (7) except (0,0).

1. Liapunov A.M. Problème général de la stabilité du mouvement. Ann. Fac. Sci. Toulouse, 9: 203–474, 1907.
2. Chetaev N.G. The stability of motion. New York, 1961.
3. Krassovskii N.N. Certain problems of the theory of stability of motion transl. Stanford, 1963.
4. Nemytskii V.V., Stepanov V.V. Qualitative Theory of Differential Equations (Russian). Moskow; Leningrad, 1st ed. 1947, 2nd ed. 1949. English translation. Prinston, 1960.
5. Hahn W. Stability of motion. Berlin, 1967.
6. Hahn W. // Math. Annalen. 1958. Vol. 136. P. 430–441.
7. Hahn W. Theorie und Anwendung der direkten methode von Ljapunov. Berlin; Gottingen; Heidelberg, 1959.
8. Halanay A., Wexler D. Teoria calitativâ a sistemelor cu impulsuri. Editura Academiei Republicii Socialiste România. Bucuresti, 1968.
9. Bulgakov N.G., Kalitine B.S., Pokataev A.V. // Minsk, 1979. Dep in VINITI 26.04.79, 1543-79.
10. Bulgakov N.G. // Vestnik Beloruss. Gos. Univ. Ser. 1. 1983. № 1. P. 45–57.
11. Rusev E.R. The dissertation on competition of a scientific degree of the candidate of mathematical sciences. Plovdiv (Bulgarian), 1987.
12. Alimov U.I. // Siberian mathematical journal. 1961. T. 2. № 1.

Поступила в редакцию 19.09.07.

Борис Сергеевич Калитин – кандидат физико-математических наук, доцент кафедры методов оптимального управления.

Жан-Клод Вивальда – доктор физико-математических наук, научный руководитель лаборатории Projet CONGE, INRIA, г. Мец, Франция.