PROCESSOR SHARING QUEUEING SYSTEM WITH LIMITED MEMORY SPACE

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Processor sharing queueing system is considered in which each customer has some random space requirement (volume), service time of the customer depends on his volume, and the total customers capacity in the system is limited by the value V (memory capacity). We find a stationary distribution of the number of customers in the system, as well as the stationary loss probability. Some particular cases are analyzed.

Key words: processor sharing, space requirement, memory capacity, volume of a customer, total volume of customers.

Processor sharing queueing models have been used to solve various problems occurring in computer and communicating systems designing. Presently, they are applicable to situations where a common recourse is shared by a varying number of concurrent users [1] (for example, to WEB – servers modeling [2]). Let ξ be the length of the customer, i.e. ξ is a customer service time under condition that there are no other customers in the system during the customer service. Later we shall also use a notion of residual length of the customer. This is a rest of service time after some time moment t under condition that there are no other customers in the system during the time of service termination of the customer.

We consider the system that differs from the classical $M/G/1-EPS$ system [3] in the following properties.

1. Each customer, independently of his arrival time and characteristics of other customers, is characterized by the random space requirement ζ that is called the volume of the customer.

2. Random variables ζ and ξ are generally dependent. The distribution function $F(x,t)=\mathbf{P}\{\zeta<x,\xi<t\}$ is given.

3. The total volume of customers (i.e. the total sum $\sigma(t)$ of space requirements of customers present in the system at arbitrary time moment t) is limited by the value $V>0$, which is called the memory capacity.

Denote by $L(x)=F(x,\infty)$ and $B(t)=F(\infty,t)$ the distribution functions of the random variables ζ and ξ respectively. The customers total volume limitation leads to their losses. A customer having the space requirement x who arrives at the epoch τ will be admitted to the system if $\sigma(\tau-0)+x\le V$. Otherwise ($\sigma(\tau-0)+x>V$), the customer will be lost. Note that the case of $V=\infty$ was analyzed in the papers [4, 5].

In the present paper we obtain the relations for stationary distribution of number of customers present in the system and the relations for loss probability. Note that for the case of $V<\infty$ the stationary mode exists for the system under consideration, if $\rho=a\beta_1<\infty$, where a is a parameter of customers entrance flow.

Let $\eta(t)$ be a number of customers present in the system under consideration at time moment t. Assume that customers in the considered system at an arbitrary time instant t are enumerated at random; i.e., if the number of customers is k, then there are $k!$ ways to enumerate them, and each enumeration can be chosen with the same probability $1/k!$. Denote by $\sigma_j(t)$ the space requirement of jth customer present in the system at time moment t, let $\xi_j^*(t)$ be the residual length of the customer at time moment t, $j=\overline{1,\eta(t)}$. Then the system under consideration is described by the Markov random process

$$(\eta(t); \sigma_j(t), \xi_j^*(t), j=\overline{1,\eta(t)}), \tag{1}$$

Note that in our notations we have $\sigma(t)=\sum_{j=1}^{\eta(t)}\sigma_j(t)$. In what follows, to simplify the notations we denote

$$Y_k=(y_1,\ldots,y_k),\ Y_k^j=(y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_k),\ Y_k^j(u)=(y_1,\ldots,y_{j-1},u,y_{j+1},\ldots,y_k).$$

Sometimes, in the case $k=1$, instead of Y_1 we write y_1 or the value that this component takes, and in the case $k=2$, instead of Y_2 we write (y_1,y_2) or their values. In other words, we sometimes specify vectors of small dimensions by indicating their components. We shall characterize the process (1) by the functions with the following probabilistic sense:

$$G_k(x,Y_k,t)=\mathbf{P}\{\eta(t)=k;\sigma(t)<x,\xi_j^*(t)<y_j,\ j=\overline{1,k}\},\ \ k=1,2,\ldots; \tag{2}$$

$$\Theta_k(Y_k,t)=\mathbf{P}\{\eta(t)=k;\xi_j^*(t)<y_j,\ j=\overline{1,k}\}=G_k(V,Y_k,t),\ \ k=1,2,\ldots; \tag{3}$$

$$P_0(t)=\mathbf{P}\{\eta(t)=0\}; \tag{4}$$

$$P_k(t)=\mathbf{P}\{\eta(t)=k\}=G_k(V,\infty_k,t)=\Theta_k(\infty_k,t),\ \ k=1,2,\ldots, \tag{5}$$

where $\infty_k=(\infty,\ldots,\infty)$ is a k-dimensional vector.

If the condition $\rho=a\beta_1<\infty$ is satisfied, then $\eta(t)\Rightarrow\eta$, $\sigma(t)\Rightarrow\sigma$, and $\xi_j^*(t)\Rightarrow\xi_j^*$ in the weak convergence sense; i.e. the following limits exist:

$$g_k(x,Y_k)=\lim_{t\to\infty}G_k(x,Y_k,t)=\mathbf{P}\{\eta=k;\sigma<x,\xi_j^*<y_j,\ j=\overline{1,k}\},\ k=1,2,\ldots; \tag{6}$$

$$\theta_k(Y_k)=\lim_{t\to\infty}\Theta_k(Y_k,t)=\mathbf{P}\{\eta=k;\xi_j^*<y_j,\ j=\overline{1,k}\}=g_k(V,Y_k),\ k=1,2,\ldots; \tag{7}$$

$$p_0=\lim_{t\to\infty}P_0(t)=\mathbf{P}\{\eta=0\}; \tag{8}$$

$$p_k=\lim_{t\to\infty}P_k(t)=\mathbf{P}\{\eta=k\}=g_k(V,\infty_k)=\theta_k(\infty_k),\ k=1,2,\ldots. \tag{9}$$

Note that the functions $g_k(x,Y_k)$ and $\theta_k(Y_k)$ are symmetric with respect to permutations of components of the vector Y_k due to our random enumeration of customers in the system.

Using the method of auxiliary variables [6], we can write out the partial differential equations for the functions defined by (2)–(4). Then, passing to the limit as $t\to\infty$ in these equations, we obtain the following stationary equations for functions (6)–(8):

$$0=-ap_0L(V)+\left.\frac{\partial\theta_1(y)}{\partial y}\right|_{y=0}; \tag{10}$$

$$-\frac{\partial\theta_1(y)}{\partial y}+\left.\frac{\partial\theta_1(y)}{\partial y}\right|_{y=0}=ap_0F(V,y)-a\int_0^V g_1(V-x,y)dL(x)+$$

$$+\frac{1}{2}\left[\left.\frac{\partial\theta_2(y,u)}{\partial u}\right|_{u=0}+\left.\frac{\partial\theta_2(u,y)}{\partial u}\right|_{u=0}\right]; \tag{11}$$

$$-\frac{1}{k}\sum_{j=1}^{k}\left[\frac{\partial\theta_k(Y_k)}{\partial y_j}-\left.\frac{\partial\theta_k(Y_k)}{\partial y_j}\right|_{y_j=0}\right]=\frac{a}{k}\sum_{j=1}^{k}\int_0^V g_{k-1}(V-x,Y_k^j)dF_x(x,y_j)-$$

$$-a\int_0^V g_k(V-x,Y_k)dL(x)+\frac{1}{k+1}\sum_{j=1}^{k+1}\left.\frac{\partial\theta_{k+1}(Y_{k+1}^j(u))}{\partial u}\right|_{u=0},\ k=2,3,\ldots. \tag{12}$$

In stationary mode, we have boundary conditions described by the following equilibrium equations:

$$a\int_0^V g_k(V-x,Y_k)dL(x)=\frac{1}{k+1}\sum_{j=1}^{k+1}\left.\frac{\partial\theta_{k+1}(Y_{k+1}^j(u))}{\partial u}\right|_{u=0},\ k=1,2,\ldots. \tag{13}$$

To write out the solution of eq. (10)–(12) that satisfy to (13) and the normalization condition, we introduce the function having the following probabilistic sense:

$$H(x,y)=\mathbf{P}\{\zeta<x,\xi\geq y\}=\mathbf{P}\{\zeta<x\}-\mathbf{P}\{\zeta<x,\xi<y\}=L(x)-F(x,y). \tag{14}$$

Let us also introduce the function $\Phi_y(x)=\int_0^y H(x,u)du$. Its meaning becomes obvious if we use the representation $F(x,u)=L(x)B(u|\zeta<x)$, where $B(u|\zeta<x)=\mathbf{P}\{\xi<u|\zeta<x\}$ is the conditional distribution function of customer length given that his volume is less than x. Then (14) implies

$$\Phi_y(x)=L(x)\int_0^y[1-B(u|\zeta<x)]du. \tag{15}$$

Let us also introduce the following notation for Stieltjes convolution:

$$F_1*\ldots*F_n(x)=\mathop{*}_{j=1}^{n}F_j(x).$$

Then, using the above–mentioned symmetry property of functions (6) and (7) and taking into account boundary conditions (13), one can show by direct substitution that the solution of eq. (10)–(12) can be represented as

$$g_k(x,Y_k)=Ca^k\mathop{*}_{j=1}^{k}\Phi_{y_j}(x),\ k=1,2,\ldots, \tag{16}$$

where C is a constant to be specified later from the normalization condition.

Introduce the notation $R(x)=\int_{u=0}^{x}\int_{y=0}^{\infty}ydF(u,y)$. The function $R(x)$ has the meaning of "partial" mathematical expectation [7] of the random variable ξ with respect to the event $\{\zeta<x\}$, i.e. $R(x)=\mathbf{E}(\xi,\zeta<x)=\mathbf{E}(\xi|\zeta<x)L(x)$, where $\mathbf{E}(\xi|\zeta<x)$ is the conditional mathematical expectation of the customer length given that his volume is less than x. It is easily seen that

$$R(x)=\lim_{y\to\infty}\Phi_y(x)=\int_0^\infty H(x,u)du = L(x)\int_0^\infty [1-B(u|\zeta<x)]du. \qquad (17)$$

Using relation (7), we obtain

$$\theta_k(Y_k)=Ca^k \mathop{*}_{i=1}^{k} \Phi_{y_i}(V),\ k=1,2,\ldots.$$

It follows from (9) that

$$p_k = Ca^k R_*^{(k)}(V),\ k=1,2,\ldots, \qquad (18)$$

where $R_*^{(k)}(V)$ is the value of kth Stieltjes convolution of the function $R(x)$ in the point of V. From the normalization condition $\sum_{k=0}^{\infty} p_k = 1$ we obtain

$$C=p_0=\left(\sum_{k=0}^{\infty} a^k R_*^{(k)}(V)\right)^{-1}. \qquad (19)$$

Finding the stationary loss probability P is based on the fact that in stationary mode the average number of customers accepted for service within a time unit (i.e. customers who entered the system during this time period and were not lost) is equal to the average number of customers whose service was terminated within this time period. Thus, taking into account the symmetry of $\theta_k(Y_k)$ with respect to the above–mentioned permutations of components of the vector Y_k, we obtain the following equilibrium equation:

$$a(1-P)=\sum_{j=1}^{\infty}\left.\frac{\partial\theta_j(\infty_{j-1},u)}{\partial u}\right|_{u=0}.$$

Taking into account that $\left.\frac{\partial\Phi_u(x)}{\partial u}\right|_{u=0} = L(x)$, we have

$$P=1-p_0\sum_{k=0}^{\infty} a^k L * R_*^{(k)}(V). \qquad (20)$$

If random variables ζ and ξ are independent (i.e. $F(x,t)=L(x)B(t)$), we have $R(x)=\beta_1 L(x)$, where $\beta_1=\int_0^\infty [1-B(x)]dx$ is the first moment of customer length. So, relations (18)–(20) take the following form:

$$p_0=\left[\sum_{k=0}^{\infty}\rho^k L_*^{(k)}(V)\right]^{-1},\ p_k=p_0\rho^k L_*^{(k)}(V),\ k=1,2,\ldots,\ P=1-p_0\sum_{k=0}^{\infty}\rho^k L_*^{(k+1)}(V).$$

Let us analyze two particular cases.

1. Discrete case. Assume that each customer's demand consists of random number of symbols. Let $q_i=\mathbf{P}\{\zeta=i\}$, $i=1,2,\ldots,$ $\sum_{i=1}^{\infty} q_i = 1$, be the probabilities that a customer volume is equal to i. For example, an arbitrary customer needs during his service for i units of homogeneous resource with probability q_i; and the total number of units is limited by $V=N$, $N=1,2,\ldots$. If the customer needs for i units of resource, we shall call him i-customer, or the customer of ith type. Denote by $B_i(t)=\mathbf{P}\{\xi<t|\zeta=i\}$ the distribution function of service time of

i-customer. Obviously, we have $F(x,t)=\sum_{i<x} q_i B_i(t)$, $L(x)=\sum_{i<x} q_i$, $B(t)=\sum_{i=1}^{\infty} q_i B_i(t)$. The function (15) in this case takes the form $\Phi_y(x)=\int_0^y H(x,u)\,du=$

$=\int_0^y [L(x)-F(x,u)]\,du=\sum_{i<x} q_i \int_0^y [1-B_i(u)]\,du$; and the function $R(x)$ can be obtained from (17) as $R(x)=\sum_{i<x} q_i \beta_{i1}$, where $\beta_{i1}=\int_0^{\infty}[1-B_i(u)]\,du$ is the first moment of the length of i-customer. From the relations (18), (19) we obtain the following formulas for stationary customers number distribution:

$$p_k=p_0 a^k \sum_{i_1+\ldots+i_k\le N} \prod_{j=1}^{k} q_{i_j}\beta_{i_j 1},\ k=\overline{1,N},\ p_0=\left[1+\sum_{k=1}^{N} a^k \sum_{i_1+\ldots+i_k\le N} \prod_{j=1}^{k} q_{i_j}\beta_{i_j 1}\right]^{-1},$$

where $i_1,\ldots,i_k$ is the type of the 1st, 2nd,…, kth customer accordingly.

From (20) we have for the loss probability:

$$P=1-p_0\sum_{i=1}^{N} q_i \sum_{k=0}^{N-1} a^k \sum_{i_1+\ldots+i_k\le N-i} \prod_{j=1}^{k} q_{i_j}\beta_{i_j 1}.$$

2. The customer length is proportional to his volume. Let customer volume ζ has an exponential distribution with the parameter f and his length $\xi=c\zeta,\ c>0$. Then we obtain after some calculations:

$$p_0=\frac{1-\rho}{1-\sqrt{\rho}e^{-fV}[\mathrm{sh}(\sqrt{\rho}fV)+\sqrt{\rho}\,\mathrm{ch}(\sqrt{\rho}fV)]},\ p_k=p_0\rho^k\left[1-e^{-fV}\sum_{i=0}^{2k-1}\frac{(fV)^i}{i!}\right],\ k=1,2,\ldots,$$

$$P=p_0 e^{-fV}\mathrm{ch}(\sqrt{\rho}fV).$$

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