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SUBDIFFERENTIAL SLOPES AND STABILITY OF ERROR BOUNDS FOR CONVEX CONSTRAINT SYSTEMS

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1. Error Bounds. Given a function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a Banach space X and a point $\bar{x} \in X$ with $f(\bar{x}) = 0$, we say that f admits a (local) *error bound* at $\bar{x}$ if there exist reals $c > 0$ and $\delta > 0$ such that

$$cd(x, S_f) \le [f(x)]_+ \quad \text{for all } x \in B_\delta(\bar{x}),$$

where $S_f := \{x \in X : \ f(x) \le 0\}$ and the notation $\alpha_+ := \max(\alpha, 0)$ is used, or equivalently

$$\operatorname{Er} f(\bar{x}) := \liminf_{x \to \bar{x},\, f(x) > 0} \frac{f(x)}{d(x, S(f))} > 0.$$

2. Subdifferential Slopes. From now on, $f : X \to \mathbb{R}_\infty$ is a proper lower semicontinuous convex function on a Banach space X and $f(\bar{x}) < \infty$. Recall the definition of the *subdifferential* of f at $\bar{x}$:

$$\partial f(\bar{x}) = \left\{x^* \in X^* |\ f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle,\ \forall x \in X\right\}.$$

The *subdifferential slope*, *boundary subdifferential slope*, and *strict outer subdifferential slope* of f at $\bar{x}$ are defined as follows:

$$|\partial f|(\bar{x}) = \inf\{\|x^*\| \mid x^* \in \partial f(\bar{x})\},$$
$$|\partial f|_{bd}(\bar{x}) = \inf\{\|x^*\| \mid x^* \in \operatorname{bd} \partial f(\bar{x})\},$$
$$\overline{|\partial f|}^{>}(\bar{x}) = \liminf_{x \to \bar{x},\, f(x) \downarrow f(\bar{x})} |\partial f|(x).$$

Proposition 1. $|\partial f|(\bar{x}) \le |\partial f|_{bd}(\bar{x}) \le \overline{|\partial f|}^{>}(\bar{x})$.

All inequalities in the above proposition can be strict.

3. Stability of Local Error Bounds. Here we assume that $f(\bar{x}) = 0$.

Theorem 1. $\operatorname{Er} f(\bar{x}) = \overline{|\partial f|}^{>}(\bar{x})$.

Corollary 1. *Consider the following properties:*

(i) f admits an error bound at $\bar{x}$;

(ii) $\overline{|\partial f|}^{>}(\bar{x}) > 0$; *(iii)* $|\partial f|_{bd}(\bar{x}) > 0$; *(iv)* $0 \notin \partial f(\bar{x})$; *(v)* $0 \in \operatorname{int} \partial f(\bar{x})$.

Each of the properties (ii)–(v) is sufficient for the error bound property (i). Moreover,
[(iv) or (v)] $\Leftrightarrow$ *(iii)* $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(i).*

Definition 1. *Let $\varepsilon \ge 0$. We say that a convex and lower semicontinuous function $g : X \to \mathbb{R}_\infty$ is an ε-*perturbation *of f near $\bar{x}$ and write $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ if $g(\bar{x}) = f(\bar{x})$ and*

$$\limsup_{x \to \bar{x}} \frac{|g(x) - f(x)|}{\|x - \bar{x}\|} \le \varepsilon$$

with the convention $\infty - \infty = 0$.

Proposition 2. *If $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$, then $f \in \operatorname{Ptb}(g, \bar{x}, \varepsilon)$, $\partial g(\bar{x}) \subseteq \partial f(\bar{x}) + \varepsilon \mathbb{B}^*$, $|\partial g|(\bar{x}) \ge |\partial f|(\bar{x}) - \varepsilon$, and $|\partial g|_{bd}(\bar{x}) \ge |\partial f|_{bd}(\bar{x}) - \varepsilon$.*

The next theorem shows that condition (iii) in Corollary 1 provides a characterization of the "combined" error bound property for the family of ε-perturbations of f near $\bar{x}$.

Theorem 2. *Let $\varepsilon > 0$. The following assertions hold true:*

(i) $\operatorname{Er} g(\bar{x}) \ge |\partial f|_{bd}(\bar{x}) - \varepsilon$ *for any* $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$;

(ii) if $0 \in \operatorname{bd} \partial f(\bar{x})$, *then* $\operatorname{Er} g(\bar{x}) \le \varepsilon$ *where*

$$g(x) := f(x) + \varepsilon \|x - \bar{x}\|, \quad x \in X;$$

(iii) if $\dim X < \infty$ *and* $0 \in \operatorname{bd} \partial f(\bar{x})$, *then there exists an* $x^* \in \varepsilon B^*$ *such that* $\operatorname{Er} g(\bar{x}) \le \varepsilon$ *where*

$$g(x) := f(x) + \langle x^*, x - \bar{x} \rangle, \quad x \in X.$$

Given a number $\varepsilon \geq 0$, denote

$$\mathrm{Er}\,\{\mathrm{Ptb}\,(f,\bar{x},\varepsilon)\}(\bar{x}) := \inf_{g\in\mathrm{Ptb}\,(f,\bar{x},\varepsilon)} \mathrm{Er}\,g(\bar{x}).$$

This number characterizes the error bound property for the whole family of convex ε-perturbations of f near $\bar{x}$. Obviously, $\mathrm{Er}\,\{\mathrm{Ptb}\,(f,\bar{x},\varepsilon)\}(\bar{x}) \leq \mathrm{Er}\,f(\bar{x})$.

Corollary 2. *The following properties are equivalent:*

(i) $\mathrm{Er}\,\{\mathrm{Ptb}\,(f,\bar{x},\varepsilon)\}(\bar{x}) > 0$ *for some* $\varepsilon > 0$*;*
(ii) $0 \notin \mathrm{bd}\,\partial f(\bar{x})$.

For a detailed study of stability of local and global error bounds for convex functions and convex semi-infinite constraint systems of the form

$$f_t(x) \leq 0 \quad \text{for all} \quad t \in T,$$

where T is a compact Hausdorff space, $f_t : X \to \mathbb{R}$, $t \in T$, are given continuous convex functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in X$, we refer the reader to [1, 2].

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DAD SYSTEMS OF CONTROL AND OBSERVATION

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Scientific-technical progress, in particular, the widespread use of microprocessors in industry, subjects the development of new control and observation systems and the support of existing ones to new, higher requirements related to the necessity of a more adequate description of these systems and the use of their specific properties, which often leads to hybrid dynamical systems.