MULTIFACTOR MODELS OF TERM STRUCTURE OF YIELD FOR ZERO COUPON BONDS

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Abstract

Properties of such characteristics of term structure of interest rates as yield curve and forward rates in a case when the affine model of yield is used are discussed. Unlike known approaches are analyzed not only one-factor, but also multifactor models. Be-sides, it is considered not only a range short terms and mean terms to maturity of as-sets, but also long terms. It is offered to use a duration riskless rates as a time variable. It gives the possibility to compare the yield and forward curves on all interval of change of terms to maturity of assets.

Let's consider the nominal bonds that sold by auction on some current time t at the price of $P(t, T, x)$, where T is the bond maturity date, and $x = x(t)$ is generally a vector of the variables characterizing a state of the financial market on date t, $t < T$. It is considered that the bond is free from non-payment and on date T is repaid for 1 monetary unit, i.e. the price $P(T, T, x) = 1$ for any states $x(T)$.

As the yield interest rate to maturity (or simply yield) is called the quantity

$$y(t, T, x) = \frac{-\ln P(t, T, x)}{T - t}.$$

Term structure of yield name the dependence $y(t, T, x)$ on term to maturity $T - t$. It is of interest for the investors who take care of efficiency of the investments in the future.

The short-term yield interest rate (or simply the short rate) is determined as a limit

$$y(t, x) = \lim_{T \to t} \frac{-\ln P(t, T, x)}{T - t} = \left.\frac{\partial \ln P(t, T, x)}{\partial t}\right|_{T=t}. \quad (1)$$

This rate various authors is called also as the spot rate or the risk-free rate as it characterizes yield during infinitesimal time interval on which how it changed the market state, it "will not have time" to become risky.

Along with the yield rate to maturity, that characterizes the bond yield for all period of its activity, investors interest the bond yield on some time distance between the future dates T_1 and T_2 on the basis of the information on the yield which are available at current time t, $t < T_1 < T_2$. Such rates $f(t, T_1, T_2)$ are called as forward. Forward rates at $T_1 \to T_2 = T$ determine short-term rates for the future time point T and are called as instant forward rates $f(t, T, x)$. They interest investors is more often and a word combination "forward rates" is usual concerns to $f(t, T, x)$. The forward rate $f(t, T, x)$ is determined by a relation [5]

$$f(t, T, x) = -\frac{\partial \ln P(t, T, x)}{\partial T}.$$

There are available one-to-one ratios between the yield rate to maturity and the forward rate

$$y(t,T,x) = \frac{1}{T-t}\int_t^T f(t,s,x)ds, \;\; f(t,T,x) = y(t,T,x) + (T-t)\frac{\partial y(t,T,x)}{\partial T}.$$

The vector of states of the financial market $X(t) = (X_1, X_2, \ldots, X_n)$ follows homogeneous on time the Markov process generated by the stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t),$$

where a drift n-vector $\mu(x)$, $(n \times m)$-matrix volatility $\sigma(\xi)$ and m-vector $W(t)$ are independent standard Wiener processes.

It is supposed that function $P(t,T,x)$ of the bond prices is differentiable on the first argument and twice differentiable on the third argument. According to the stochastic analysis of Ito the bond price $P(t,T,X(t)) = Z(t)$, as time function, too is the stochastic process of diffusion type and satisfies to the stochastic differential equation

$$dZ(t) = \mu_P(t,T,x)dt + \sigma_P(t,T,x)^T dW(t),$$

where $\mu_P(t,T,x)$ and $\sigma_P(t,T,x)$ are scalar functions of drift and m-vector volatility, accordingly, determined by ratios

$$\mu_P(t,T,x) = \frac{\partial P(t,T,x)}{\partial t} + \mu(x)^T\frac{\partial P(t,T,x)}{\partial x} + \frac{1}{2}\mathrm{tr}\left(\sigma(x)^T\frac{\partial^2 P(t,T,x)}{\partial x^2}\sigma(x)\right),$$

$$\sigma_P(t,T,x)^T = \sigma(x)^T\frac{\partial P(t,T,x)}{\partial x}.$$

The equation for determination of function $P(t,T,x)$ is derived from condition of absence of arbitrage opportunities in the financial market [6] which in the case under consideration multifactor model is reduced to that there should be an m-vector $\lambda(t,x)$, not dependent on repayment date of bonds T, such that the equality $\mu_P(t,T,x) - y(x)P(t,T,x) = \sigma_P(t,T,x)^T\lambda(t,x)$ was carried out. Function $\lambda(t,x)$ is called as risk market price. Thus, we come to the equation with partial derivatives for function $P(t,T,x)$:

$$\frac{\partial P(t,T,x)}{\partial t} + \mu(x)^T\frac{\partial P(t,T,x)}{\partial x} + \frac{1}{2}\mathrm{tr}\left(\sigma(x)^T\frac{\partial^2 P(t,T,x)}{\partial x^2}\sigma(x)\right) - y(x)P(t,T,x) =$$

$$= \lambda(t,x)^T\sigma(x)^T\frac{\partial P(t,T,x)}{\partial x}.$$

This equation should be solved with boundary condition $P(T,T,x) = 1$ for any states x.

That fact that Markov process $X(t)$ is homogeneous for time, leads to following property of function $P(t,T,x)$: it depends not on t and T separately, but only on difference $T-t$, i.e. not on current time and maturity date, but only on a remaining period to maturity $\tau = T-t$. So $P(t,T,x) \leftrightarrow P(\tau,x)$, thus $y(t,x) \leftrightarrow y(x)$, $\lambda(t,x) \leftrightarrow$

$\lambda(x)$. In the literature often assume that a drift vector $\mu(x)$ and a diffusion matrix $\sigma(x)\sigma(\xi)^T$ of states of the financial market are described by affine functions, and risk market prices are such that $\sigma(x)\lambda(x)$ is n-vector with affine components,

$$\mu(x) = K(\theta - x),\ \sigma(x)\sigma(x)^T = \alpha + \sum_{i=1}^{n} \beta_i x_i,\ \sigma(x)\lambda(x) = \xi + \sum_{i=1}^{n} \eta_i x_i.$$

Here K, α and β_i are $(n \times n)$-matrices; θ, ξ and η_i are n-vectors, x_i are the components of the vector x. Note that these relations are satisfied at

$$\sigma(x) = \sigma \left\langle \sqrt{\gamma + \Gamma x} \right\rangle,\ \lambda(x) = \left\langle \sqrt{\gamma + \Gamma x} \right\rangle \lambda,$$

where γ and λ are m-vectors, σ and Γ are respectively $(n \times m)$- and $(m \times n)$-matrices, and $\langle \sqrt{\gamma + \Gamma x} \rangle$ is a diagonal $(m \times m)$-matrix on which diagonal there are square roots of components of vector $\gamma + \Gamma x$. In this case $\alpha = \sigma\langle\gamma\rangle\sigma^T$, $\xi = \sigma\langle\gamma\rangle\lambda$, and the matrix elements β_i and the vector η_i are determined by equalities

$$(\beta_i)_{kj} = \sum_{u=1}^{m} \sigma_{ku}\sigma_{ju}\Gamma_{ui},\ 1 \le k,\ j \le n;\ (\eta_i)_k = \sum_{u=1}^{m} \sigma_{ku}\Gamma_{ui}\lambda_u,\ 1 \le k \le n.$$

Such assumptions lead to affine time structure of yield interest rates. We will rewrite the equation for the price of bond $P(t, T, x)$ in this case

$$-\frac{\partial P(\tau, x)}{\partial \tau} + (\theta - x)^T K^T \frac{\partial P(\tau, x)}{\partial x} + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 P(\tau, x)}{\partial x^2}\left(\alpha + \sum_{i=1}^{n} \beta_i x_i\right)\right) - y(x)P(\tau, x) =$$

$$= \left(\xi + \sum_{i=1}^{n} \eta_i x_i\right)^T \frac{\partial P(\tau, x)}{\partial x}. \tag{2}$$

The solution of this equation can be presented in the form

$$P(\tau, x) = \exp\left\{A(\tau) - x^T B(\tau)\right\},$$

where the functions $A(\tau)$ and $B(\tau)$ satisfy the entry conditions: $A(0) = 0$, $B(0) = 0$. Note that for the bond price in such representation the short-term interest rate (1) takes a form

$$y(x) = \lim_{\tau \to 0} \frac{-\ln P(\tau, x)}{\tau} = \lim_{\tau \to 0} \frac{x^T B(\tau) - A(\tau)}{\tau} = x^T B'(0) - A'(0), \tag{3}$$

i. e. $y(x)$ is an affine function of a vector x. The prime designates a derivative on τ. Note that the state of the financial market is usual characterized by values of interest rates, in other words, vector components x are the quantities, making sense the interest rates. When interest rates are equal to zero, the bond return is absent, therefore in expression (3) it is necessary to put $A'(0) = 0$. Further this assumption will be accepted in all cases. We will designate $B'(0) = \phi$. The vector ϕ can be considered as a vector of weight coefficients which correspond to this or that a component of a state vector x under short-term rate determination $y(x) = x\phi = x_1\phi_1 + x_2\phi_2 + \cdots + x_n\phi_n$, $1 \le i \le n$. If there is the risk-free rate r among components of state vector x (assume $x_1 = r$) then by definition $y(x) = x\phi = r$, and other components of the vector $B'(0) = \phi$ have to be equal to zero, i.e. $\phi_1 = 1$, $\phi_i = B_i'(0) = 0$, $2 \le i \le n$.

Substitution of solution $P(\tau, x) = \exp\left\{A(\tau) - x^T B(\tau)\right\}$ in the equation (2) for $P(t, T, x)$ leads to the ordinary differential equations for function $A(\tau)$ and a component of vector $B(\tau) = (B_1(\tau), B_2(\tau), \ldots, B_n(\tau))$:

$$A'(\tau) = (\xi - K\theta)^T B(\tau) + B(\tau)^T \alpha B(\tau)/2,\ A(0) = 0, \tag{4}$$

$$B_i'(\tau) = \phi_i - B(\tau)^T(\eta_i + K_i) - B(\tau)^T \beta_i B(\tau)/2,\ B_i(0) = 0. \tag{5}$$

In the equation for $B_i(\tau)$ symbol K_i designates i-th column of matrix K, $1 \le i \le n$.

Note that it follows from the definitions above, within the frameworks of affine structure the yield rate and the forward rate are determined by expressions

$$y(\tau, x) = \frac{x^T B(\tau) - A(\tau)}{\tau},\ f(\tau, x) = x^T \frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}. \tag{6}$$

$$y(\tau, x) = \frac{1}{\tau}\int_0^\tau f(s, x)ds,\ f(\tau, x) = y(\tau, x) + \tau\frac{\partial y(\tau, x)}{\partial \tau}. \tag{7}$$

Functions $y(\tau, x)$ and $f(\tau, x)$, considered how functions from a variable τ, is usual are called by the yield curve and the forward curve respectively. A forms and properties of these functions are of interest for investors. Therefore further we will be interested in explicit analytical expression of these functions and determination of their properties. We will find out in the beginning some general properties.

General limit of both curves on the left end, i.e. when $\tau \to 0$, is the short-term interest rate of yield $y(x)$. Really, as $A'(0) = 0$ and $B'(0) = \phi$,

$$y(x) = x^T\phi = \lim_{\tau \to 0} y(\tau, x) = \lim_{\tau \to 0} f(\tau, x).$$

For small terms to maturity according to (7) representations are valid

$$y(\tau, x) = y(x) + \tau\left(\frac{\partial y(\tau, x)}{\partial \tau}\right)_{\tau=0} + o(\tau),\ f(\tau, x) = y(x) + 2\tau\left(\frac{\partial y(\tau, x)}{\partial \tau}\right)_{\tau=0} + o(\tau).$$

From here it is visible that starting from one point $y(x)$, curves $y(\tau, x)$ and $f(\tau, x)$ disperse with growth τ. In addition the forward curve changes twice faster.

If the yield curve has an extremum for some term to maturity τ_*, i.e. $\left.\frac{\partial y(\tau,x)}{\partial \tau}\right|_{\tau=\tau_*} = 0$ then the forward rate and the yield rate for this term τ_* coincides with the value $f(\tau_*, x) = y(\tau_*, x)$.

From this the general conclusion that if the yield curve has a maximum (minimum) $y(\tau_*, x)$ the greatest (least) value of the forward rate f^* always is more (less) than such value $y(\tau_*, x)$ follows.

For that finding-out as curves $y(\tau, x)$ and $f(\tau, x)$ behave for long terms to maturity, it is required to know properties of functions $A(\tau)$ and $B(\tau)$. However to solve the equations for these functions generally it is not possible. It is possible to tell only that vector $B(\tau)$ is the solution of the multivariate Riccati equation. If it is succeeded in finding the vector $B(\tau)$ then the function $A(\tau)$ is simply integration of the right hand side of equation for $A(\tau)$. More detailed properties of functions $A(\tau)$ and $B(\tau)$ can be found out only when the concrete values of parameters K, α, β, η, θ and ξ are known. However using some expected properties of functions $A(\tau)$ and $B(\tau)$ it is possible to find out expected properties of curves $y(\tau, x)$ and $f(\tau, x)$.

The state of the financial market can be presented by a set of yield rates of various securities, therefore, components of a vector x can make sense of interest rates [3]. In other cases the state of the financial market can be described by an interest rate and its volatility [4]. Because change of an interest rate and volatility differently influence on yield to maturity, signs a component of a vector $B(\tau)$ corresponding to them, have to be coordinated with these components of a vector of a state x. With growth of a risk-free rate the yield to maturity (6) increases, therefore, if x_1 is an interest rate, then $B_1(\tau) > 0$. With growth of volatility the yield to maturity (6) decreases, therefore, if x_k is volatility, then $B_k(\tau) < 0$.

It is clear also that with increase in term to maturity $\tau = T - t$ influence of a state of the market $x \equiv X(t)$ in timepoint t on yield to maturity (6) has to decrease and in a limit at $\tau \to +\infty$ to disappear at all, that is $\lim_{\tau\to\infty} x^T B(\tau)/\tau = 0$. It gives the grounds to believe that limits of $B_k(\tau)$ at $\tau \to +\infty$ exist and $\lim_{\tau\to\infty} B(\tau) = B(\infty)$, $\|B(\infty)\| < \infty$. In these conditions it is natural to expect that $\lim_{\tau\to\infty} B'(\tau) = 0$. In this case the vector $B(\infty)$ according to (5) can be defined from system of the equations

$$\phi_i = B(\infty)^T(\eta_i + K_i) + B(\infty)^T \beta_i B(\infty)/2, \ 1 \le i \le n.$$

Riccati equation (5) in the one-dimensional case determines the solution in the form of monotone function. When factor x is the interest rate, the function $B(\tau)$ increase from $B(0) = 0$ up to $B(\infty) > 0$, i.e. $0 \le B(\tau) \le B(\infty)$. For the case, when x is volatility, the function $B(\tau)$ decrease from $B(0) = 0$ down to $B(\infty) < 0$, i.e. $0 \ge B(\tau) \ge B(\infty)$. Thus, it is possible to expect that defined from the equations (5) components of the vector $B(\tau)$ are monotone bounded functions (numerical calculations confirm it), i.e. it is expected that $0 < |B_i(\tau)| < |B_i(\infty)|$ for all $i = 1, 2, \dots, n$, $\lim_{\tau\to\infty} B(\tau) = B(\infty)$, $\|B(\infty)\| < \infty$.

In this case the following equalities take place:

$$\lim_{\tau\to\infty} \frac{B(\tau)}{\tau} = \lim_{\tau\to\infty} \frac{dB(\tau)}{d\tau} = 0,$$

$$\lim_{\tau\to\infty} \frac{A(\tau)}{\tau} = \lim_{\tau\to\infty} \frac{dA(\tau)}{d\tau} = \left((\xi - K\theta)^T + \frac{B(\infty)^T \alpha}{2} \right) B(\infty).$$

From relation (6) also follows:

$$y(\infty, x) = f(\infty, x) = -\left((\xi - K\theta)^T + \frac{B(\infty)^T \alpha}{2} \right) B(\infty).$$

Thus, yield curve $y(\tau, x)$ and forward curve $f(\tau, x)$ coincide for short terms to maturity (at $\tau \to 0$), curves disperse with increase in τ, but at long terms (at $\tau \to \infty$) again converge to the same limit. The last property still was not specified by authors, and the yield curve $y(\tau, x)$ and the forward curve $f(\tau, x)$ are usually represented by dispersing curves. Further we will consider the concrete models which are usually considered in literature that by means of strict research to find out, whether the mentioned intuitive analysis is justified.

The component $B_1(\tau)$ of a vector $B(\tau)$, corresponding to a risk-free interest rate, has dimension of time and it can be used for measurement of duration of time as it is equal a minus of the derivative price of the bond with respect an interest rate [1, 2], i.e. duration of time measured in a certain scale. Such use is attractive because allows to show the yield curves and the forward curves for all range of terms to maturity $0 \leq \tau < \infty$ on a finite interval $0 \leq B_1(\tau) \leq B_1(\infty)$.

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