FINITE-TIME RUIN PROBABILITIES
IN THE DISCRETE RISK MODEL

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Abstract
Discrete-time risk model is considered, and recurrent algorithm for the calculating the finite-time probability of ruin is investigated. Keywords: finite-time probability of ruin, discrete time risk model.

There are a lot of papers and books devoting to the finding of the probability of ruin in different risk models. For the review see, for example, Asmussen and Albrecher [3]. In [3] it is mentioned that infinitetime ruin probabilities are obtained for the compound Poisson model with exponential claims, compound Poisson model with phase-type claim distribution, compound Poisson model with degenerate claim distribution, several special cases with heavy tails distributions. The authors above noted that exact formulas for the finite time probabilities of ruin are known in the case of Brownian motion, compound Poisson model with exponential claims and in some very specific cases. On the page 16 of [3] Asmussen and Albrecher indicated that the next after closed form solution is an alternative of obtaining exact values of the probability of ruin. In this paper we consider the recurrent algorithm for the finite-time ruin probability calculation in discrete-time risk model with discrete claims distributions. Also we consider application of this algorithm for the approximation of the ruin probability in the case of continuous claim distribution.

We consider the discrete-time risk process:

\[ U_n = u + c \cdot n - \sum_{j=1}^{n} X_j, \quad n = 1, 2, \ldots, \tag{1} \]

where \( u \) is the initial surplus, \( c > 0 \) is the constant premium rate, and \( \{X_j, j \geq 1\} \) are independent and identically distributed claim size random variables defined on the probability space \((\Omega, \mathcal{F}, P)\). For simplicity we will assume that \( u, c \) and \( X_j \) are non-negative integer, and premiums come before claims at a given time point. Denote \( P(X_j = k) = p_k, k = 0, 1, 2, \ldots \). These are standard assumptions used in the discrete time models. (See, for example, Gerber [1] and Shiu [2].) Let \( \psi_n \) denote the probability of ruin at time \( n \):

\[ \psi_n = \psi_n(u) = P(U_1 \geq 0, U_2 \geq 0, \ldots, U_{n-1} \geq 0, U_n < 0|U_0 = u). \]

Then the probability of ruin before the time \( n \) is given by the expression \( \varphi_n(u) = \sum_{k=1}^{n} \psi_k(u) \). For the probability \( \psi_n(u) \) we can obtain the following recurrent relations:

\[ \psi_1(u) = P(U_1 < 0) = P(u + c - X_1 < 0) = \sum_{k=u+c+1}^{\infty} P(X_1 = k) = 1 - \sum_{k=0}^{u+c} p_k, \tag{2} \]
\[ \psi_n(u) = \sum_{j=0}^{u+c} p_j \cdot \psi_{n-1}(c-j+u), \quad n = 2,3,\ldots \]  

We analyze the quality of the algorithm given by the relations (2-3) comparing the results with the exact ones for the case in which claim size variables are assumed to follow a geometrical distribution with probability function

\[ P(X_j = k) = p \cdot q^k, \quad k = 0, 1, 2, \ldots, \quad 0 \leq p \leq 1, \quad q = 1 - p. \]

In this special case the exact expressions are obtained by Chan and Zang in [4]. Accordingly (page 274 of [4])

\[ \psi_1(u) = q^{u+2}, \quad \psi_2(u) = pq^{u+3}(u+2), \]

\[ \psi_{n+1}(u) = p^n q^{u+2+n} \frac{(u+2)(u+2+n+1)\ldots(u+2+2n-1)}{n!}, \quad n \geq 2. \]

The results of the calculations \( p = 0.75; c = 1 \) are shown in the Table 1: the first row was calculated using formulas (2-3), the second — using formulas (4-5). We find the same results.

<table>
<thead>
<tr>
<th>( u )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>( \varphi_20 )</td>
<td>0.111096</td>
<td>0.0370265</td>
<td>0.012339</td>
<td>0.004111</td>
<td>0.001369</td>
<td>0.000456</td>
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<td>0.000456</td>
</tr>
</tbody>
</table>

Table 1: Two ways to calculate \( \varphi_{20} \)

The Figure 1 illustrates the influence of heavy tails in the claim size distribution on the probability of ruin. Here we consider the claim size distribution of the type

\[ P(X_k = l) = \frac{(l+1)^{-m}}{\zeta(m)}, \quad l = 0, 1, 2, \ldots, \quad m > 2, \]

where \( \zeta(x) = \sum_{n=1}^{\infty} n^{-x} \) is Riemann zeta function. We take \( m = 3.17988 \) to have approximately the same mean as in the case of geometrical distribution.

Figure 1: Ruin probabilities
In the case of continuous claim distribution we use discrete approximation. The case of the exponential claim distribution with the mean $1/3$ and $c = 1$ is considered in Table 2. In this case the exact probabilities of ruin are known from [4]:

$$\psi_n(u) = \frac{[\lambda \cdot (u + cn)]^{n-1} e^{-\lambda (u + cn)} u + c}{(n-1)!}.$$ 

The accuracy of the approximation equals 0.01.

<table>
<thead>
<tr>
<th>u</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>2.11E-04</td>
<td>1.26E-05</td>
<td>7.47E-07</td>
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</tbody>
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Table 2: The case of the exponential claim distribution with the mean $\frac{1}{3}$ and $c = 1$

References


