ON STATISTICAL ESTIMATION OF PARAMETERS FOR POISSON CONDITIONAL AUTOREGRESSIVE MODEL BY SPACE-TIME DATA

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Abstract

A problem of statistical estimation of parameters for Poisson conditional autoregressive model based on space-time data is considered. An approximate regression model is proposed. Statistical estimators of parameters for both models are constructed. Results of computer experiments are presented.

1 Poisson conditional autoregressive model

Space-time models are widely used in statistical analysis of real data [1]-[2]. For example, the Neyman-Scott cluster process [1] describes the process of reproduction in demography. STCAR model was applied for modeling of bankruptcy of small and medium enterprises in some provinces of Italy [2].

We propose the Poisson conditional autoregressive model which adequately describes the dynamics of the level of disease in some regions.

Introduce the notation: (Ω, F, P) is the probability space; $S = \{1, ..., n\}$ is a set of spatial objects, n is number of spatial objects; $t \in \Im = \{1, 2, ..., T\}$ is discrete time, T is duration of the observations; $x_{s,t} \in N_0 = N \bigcup \{0\}$ is a discrete random variable at time t at s ; $F_{\bar{s},<t} = \sigma\{x(\omega, u, \tau) : u \neq s, \tau < t\} \subset F$ is the σ-algebra generated by the random variables indicated in $\{\cdot\}$; $z_{s,t} \geq 0$ is an observed (known) level of factors which influence $x_{s,t}$ at the point s at time t; $\{\varphi_k(t) : 1 \leq k \leq K_s\}$ is a given set of basis functions which determine a trend ($\varphi_1(t) \equiv 1$); $L\{\xi\}$ means the probability distribution of the random variable ξ; $E\{\xi\}$ is the expectation of the random variable ξ; $\Pi(\lambda)$ is the Poisson probability distribution with the parameter $\lambda > 0$; $N_1(\mu, \sigma^2)$ is the Gaussian probability distribution with the parameters μ, σ^2; $\{\xi_{s,t}\}$ are independent Gaussian random variables, $L\{\xi_{s,t}\} = N_1(0, \sigma_s^2)$.

As in [2], we construct the Poisson conditional autoregressive model for space-time data $\{x_{s,t}\}$:

$$\begin{gathered} L\{x_{s,t}|F_{\bar{s},<t}\} = \Pi(\lambda_{s,t}), \quad s \in S, t \in \Im, \quad \lambda_{s,t} = \Lambda_{s,t} e^{\xi_{s,t}}, \\ \Lambda_{s,t} = \exp\{\textstyle\sum_{i=1}^{p_s} \alpha_{si} x_{s,t-i} + \sum_{j=1}^{q_s} \beta_{sj} z_{s,t-j-1} + \sum_{k=1}^{K_s} \gamma_{sk} \varphi_k(t) + \rho_s x_{s-1,t-1}\}, \end{gathered} \tag{1}$$

where $\{\alpha_{si} : i = 1, ..., p_s\}$, $\{\beta_{sj} : j = 1, ..., q_s\}$, $\{\gamma_{sk} : k = 1, ..., K_s\}$, $\{\rho_s, \sigma_s^2 : s = 1, ..., n\}$ are parameters of the model, p_s, q_s, K_s are given natural numbers, $\rho_1 = 0$.

The problem is in statistical estimation of parameters for the model (1) based on observations $\{x_{s,t} : s \in S, t \in \Im\}$.

2 Construction of the likelihood function

Use the matrix notation:

$$\theta_s = (\alpha_{s1}, \dots, \alpha_{sp_s}, \beta_{s1}, \dots, \beta_{sq_s}, \gamma_{s1}, \dots, \gamma_{sK_s}, \rho_s, \sigma_s^2)' \in R^{p_s+q_s+K_s+2},$$
$$\theta = (\theta_1', \dots, \theta_n')' \in R^{\sum_{s=1}^{n}(p_s+q_s+K_s+2)}, \quad \xi_s = (\xi_{s,1}, \dots, \xi_{s,T})' \in R^T.$$

Theorem 1. The conditional likelihood function for the model (1) at the point $s \in S$, provided that the random vector ξ_s is fixed, has the form:

$$L_s(\theta_s|\xi_s) = \left(\prod_{t=1}^{T} \frac{\Lambda_{s,t}^{x_{s,t}}}{x_{s,t}!}\right)\left(\prod_{t=1}^{T} e^{x_{s,t}\xi_{s,t}-\Lambda_{s,t}e^{\xi_{s,t}}}\right).$$

Note that there is a problem to assign initial values $\{x_{s,t} : t = 0, -1, \dots, -P_s + 1\}$, where $P_s = max\{p_s, 1\}$. If these values are not available, we will consider a modified likelihood function:

$$\tilde{L}_s(\theta_s|\xi_s) = \left(\prod_{t=P_s+1}^{T} \frac{\Lambda_{s,t}^{x_{s,t}}}{x_{s,t}!}\right)\left(\prod_{t=P_s+1}^{T} e^{x_{s,t}\xi_{s,t}-\Lambda_{s,t}e^{\xi_{s,t}}}\right).$$

The function $L_s(\theta_s|\xi_s)$ differs from $\tilde{L}_s(\theta_s|\xi_s)$ with the finite multiplier $P\{x_{s,1}, ..., x_{s,P_s-1}|\xi_s\} \in [0, 1]$ which does not depend on T. If $T \to \infty$, the value $P\{x_{s,1}, ..., x_{s,P_s-1}|\xi_s\}$ becomes negligible. We use the log-likelihood function for statistical estimation of the parameters. In this case, if $P\{x_{s,1}, ..., x_{s,P_s-1}|\xi_s\}$ does not depend on parameters of the model, the term in the log-likelihood function, which contains $P\{x_{s,1}, ..., x_{s,P_s-1}|\xi_s\}$, will not influence the result. If $P\{x_{s,1}, ..., x_{s,P_s-1}|\xi_s\}$ depends on parameters of the model, the effect of additional term will be insignificant at $T \to \infty$.

Define a special function $f(k, a, b, \sigma^2) = E\{\xi^{2k}e^{a\xi-be^{\xi}}\}$, $L\{\xi\} = N_1(0, \sigma^2)$:

$$f(k, a, b, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \xi^{2k}e^{a\xi-be^{\xi}}e^{-\frac{\xi^2}{2\sigma^2}}d\xi, \quad a \geq 0, b \geq 0, k \geq 0. \tag{2}$$

Theorem 2. Under conditions of Theorem 1 the log-likelihood function for a set of space-time data $\{x_{s,t}, s \in S, t = 1, 2, \dots T\}$ takes the form:

$$l(\theta) = \sum_{s=1}^{n} l_s(\theta_s),$$

$$l_s(\theta_s) = \sum_{t=P_s+1}^{T} (x_{s,t} \ln \Lambda_{s,t} - \ln x_{s,t}! + \ln f(0, x_{s,t}, \Lambda_{s,t}, \sigma_s^2)).$$

3 Calculation of the maximum likelihood estimators (MLE)

Using the likelihood function from section 2, we construct the MLE. In order to find the MLE of the model parameters we need to solve the following optimization problems for each $s \in S$:

$$l_s(\theta_s) \to \max_{\theta_s}. \tag{3}$$

A necessary condition for a local maximum in (3) is:

$$\nabla_{\theta_s} l_s(\theta_s) = 0. \tag{4}$$

Let $\hat{\theta}_s$ be a solution of the equation (4), then a sufficient condition for a local maximum in (3) at the point $\hat{\theta}_s$ is the condition of negative definiteness for the matrix of the second order derivatives. We solve (4) numerically using the Newton's iterative method which has the quadratic convergence. For this method the $(k+1)$-th iteration has the form $(k = 0, 1, 2, ...)$:

$$\theta_s^{(k+1)} = \theta_s^{(k)} - (H^{(k)})^{-1} \cdot (\nabla l_s)^{(k)}, \tag{5}$$

where $(\nabla l_s)^{(k)}$ is the vector of the first order derivatives at the point $\theta_s^{(k)}$, $H^{(k)}$ is the matrix of the second order derivatives at the point $\theta_s^{(k)}$, $\theta_s^{(0)}$ is the initial value. The iterative computation stops if $||\nabla l_s(\theta_s^{(k)})|| < \varepsilon$, where $\varepsilon > 0$ is a given small quantity which determines the computation accuracy; we define $\theta_s^{(k)}$ as a solution of (4). To find the global maximum of $l_s(\theta_s)$ we apply (5) several times with different initial values, and we choose the solution of (3) with the greatest value of the likelihood function as the estimate. To specify initial value $\theta_s^{(0)}$ for (5), we construct an approximate model given below. Other vectors of initial values we generate randomly from the set $\left\{\theta : \|\theta^{(0)} - \theta\| < M\right\}$, where M is a given value.

Since $L\{x_{s,t}\} = \Pi(\lambda_{s,t})$, then $E\{x_{s,t}\} = \lambda_{s,t}$. Following this property, instead of the model (1), let us consider an approximate regression model:

$$\ln x_{s,t} = \sum_{i=1}^{p_s} \alpha_{si} x_{s,t-i} + \sum_{j=1}^{q_s} \beta_{sj} z_{s,t-j-1} + \sum_{k=1}^{K_s} \gamma_{sk} \varphi_k(t) + \rho_s x_{s-1,t-1} + \widetilde{\xi}_{s,t}, \quad s \in S, t \in \Im, \tag{6}$$

where $\{\widetilde{\xi}_{s,t}\}$ are independent Gaussian random variables, $L\{\widetilde{\xi}_{s,t}\} = N_1(0, \sigma_s^2)$. Estimators for parameters of the model (6) computed by the least squares method are taken as an initial value $\theta_s^{(0)}$ for the iterations (5).

Provided some regularity conditions, the constructed maximum likelihood estimators are consistent, effective, asymptotically normal and asymptotically unbiased [3].

4 Results of computer simulations

Computer experiments are performed on simulated data for the model (1) with the parameters:

$$n = 3, p_s = q_s = 0, K_s = 2, s \in S = \{1, 2, 3\}, \theta_1 = (2.5, -0.01, 0, 0.7)',$$

$$\theta_2 = (2.1, -0.02, -0.02, 0.6)', \theta_3 = (3.1, -0.01, -0.2, 0.5)', \varphi_1(t) = 1, \varphi_2(t) = t.$$

The mean square error of parameter estimators is evaluated by the expression:

$$r_T(\theta) = \frac{1}{F} \sum_{k=1}^{F} ||\hat{\theta}^{(k)} - \theta||^2,$$

where $\hat{\theta}^{(k)} \in R^{12}$ is the parameter estimate for the kth realization of the space-time data, $\theta \in R^{12}$ is the true value of parameters, F is the number of realizations. Figure 1 plots the dependence of $r_T(\theta)$ on the observation time T (T varies from 20 to 140, $F = 500$, an iterative algorithm (5) was used 10 times for 10 initial values $\theta^{(0)}$) for the model (1) and the model (6).

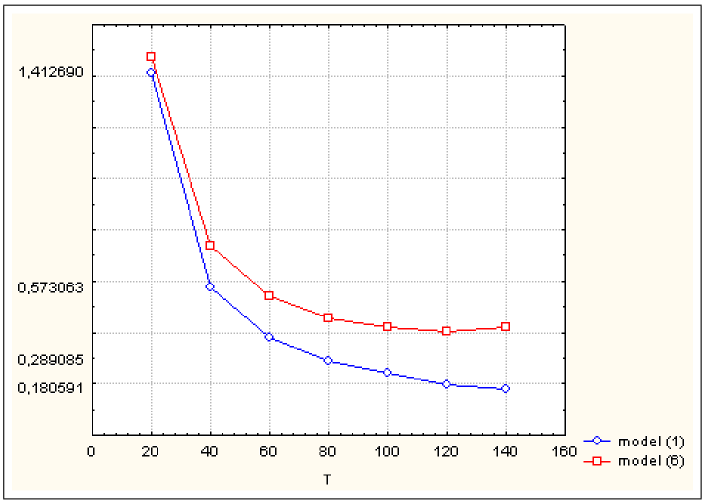

Figure 1: Mean square error for the model(1) and the model(6)

Figure 1 illustrates the property of consistency of the MLE for the model (1). We also see that the mean square error for the estimators based on the model (1) is less than for the approximate model (6).

5 Conclusion

In this paper the Poisson conditional autoregressive model based on the space-time data is constructed. The likelihood function and the MLE for this model are built. A numerical method for computation of the MLE is suggested. An approximate model is constructed. Results of computer simulation are given.

References

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