# PROPERTIES OF OPTIMAL STOPPING AND EXIT TIMES FOR DIFFUSION PROCESSES AND RANDOM WALKS

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**Abstract**

The main goal of the work is to study the limit behavior of optimal stopping and exit times for some classes of random processes, in particular Ito's diffusion, random walk and diffusion process with non-Lipschitz diffusion coefficient.

## 1 Limit behavior of optimal stopping times for Ito's diffusion

We consider an asset whose price is the solution of the following linear stochastic differential equation

$$dX_n(t) = r_n X_n(t)dt + \alpha_n X_n(t)dW(t), X_n(0) = x > 0, n \geq 0.$$

**Definition 1.** *The discounted profit is given by*

$$g_n(s, x) = e^{-\rho_n s} \cdot (x - \alpha_n).$$

**Definition 2.** *The optimal stopping time is defined as follows*

$$\tau_n^* = argmax_{\tau \in \Gamma} \left[ E^{(s,x)} e^{-\rho_n \tau} \cdot (X_n(\tau) - \alpha_n) \right],$$

*where $\Gamma$ is the set of all stopping times.*

**Definition 3.** *Optimal discounted profit is of the following form*

$$g_n^* = g_n^*(s, x) = E^{(s,x)}[g_n(\tau_n^*, X_n(\tau_n^*))].$$

We consider the case when $r_n < \rho_n$. Explicit form of $\tau_n^*$ was established, e.g. by the Oksendal in [1].

**Theorem 1.** *Let $r_n, \alpha_n, \rho_n, a_n$ converge to $r_0, \alpha_0, \rho_0, a_0$. Then the following convergence holds for every $\epsilon > 0$*

$$P(|\tau_n^* - \tau_0^*| > \epsilon) \longrightarrow 0.$$

*Moreover the stopping time $\tau_0^*$ is optimal for the limit process $X_0(t)$ with the limit parameters $r_0, \alpha_0, \rho_0, a_0$.*

The convergence of optimal discounted profit $g_n^*$ to the optimal discounted profit of the limit process also holds.

## 2 Optimal stopping problem for a random walk with polynomial reward function

For a random walk $X_t, t \in N^+$ with a drift to the left and polynomial reward function of the following form

$$g(x) = \sum_{k=1}^{n} C_k \cdot (x^+)^k, C_k \in \mathbf{R}$$

we study the optimal stopping problem of finding the optimal stopping time

$$\tau^* = \text{argmax}_{\tau \in \Gamma_0^\infty} E_x g(X_\tau) I\{\tau < \infty\}$$

using the Appel polynomials $Q_k(y)$ of the random variable $M = \sup_{k \geq 0}(X_k - X_0)$. Here $\Gamma_0^\infty$ is the set of all Markov times on $[0, \infty]$.
This method was first proposed by Novikov and Shiryaev in the work [2] for the particular case $n = 1$.

**Definition 4.** *Appel polynomials $Q_k$ of order $k \geq 0$ for random variable $M$ are defined through the expansion*

$$\frac{\exp(u \cdot y)}{E \exp(u \cdot M)} = \sum_{k=0}^{\infty} \frac{u^k}{k!} Q_k(y).$$

We establish the necessary and sufficient conditions for $C_k$ under which the linear combination of Appel polynomials associated with reward function has unique positive root $x_n$.

**Theorem 2.** *Under the conditions mentioned above the optimal stopping time is the first exit time of the random walk $X_t$ from the interval* $[-\infty, x_n]$.

Examples of random walks and reward functions which satisfy the condition of uniqueness of the positive root $x_n$ are constructed.

## 3 Asymptotics of exit times for diffusion processes

We study the process that is the solution of the following stochastich differential equation

$$X_n(t) = X_n(0) + \int_0^t b_n(s, X_n(s))ds + \int_0^t \sigma_n(s, X_n(s))dW(s), n \geq 0, t \geq 0 \qquad (1)$$

with non-random initial conditions and coefficients satisfying the following assumptions

**(a)** coefficients $b_n(s, x)$ and $\sigma_n(s, x)$ are continuous in $s$ and $x$;
**(b)** coefficients $b_n(s, x)$ and $\sigma_n(s, x)$ are of the linear growth;
**(c)** coefficient $b_n(s, x)$ satisfies the Lipschitz condition in space;
**(d)** coefficient $\sigma_n(s, x)$ satisfies the Yamada condition in space;

It was proved by Yamada [3] that under conditions **(a)** – **(d)** the stochastic differential equation (1) has unique strong solution $X_n$.

The following pointwise convergence of coefficients and initial condition is assumed

$$b_n(t,x) \to b_0(t,x), \sigma_n(t,x) \to \sigma_0(t,x), n \to \infty, \tag{2}$$

and $X_n(0) \to X_0(0), n \to \infty$.

**Theorem 3.** *Under convergence (2) the process $X_n$ converges uniformly in probability to the process $X_0$ on the arbitrary interval $[0,T]$.*

The above theorem is the generalization of result obtained in [4] on the weaker Yamada conditions.

Let $\tau_n$ and $\tau_0$ be the first exit times of the processes $X_n$ and $X_0$ from the interval $[l,r]$ where $l < X_n(0) < r, n \geq 0$.

**Theorem 4.** *Under convergence (2) the following convergence of exit times take place for every $\epsilon > 0$*

$$P(|\tau_n - \tau_0| > \epsilon) \longrightarrow 0.$$

The obtained result is useful in the investigation of the optimal stopping times convergence for diffusion processes with non-Lipschitz diffusion.

# References


[1] Oksendal B. (2003). *Stochastic Differential Equations: An Introduction with Applications*. Springer.

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[3] Ikeda N., Watanabe S. (1981). *Stochastic differential equations and diffusion processes*. North-Holland Publishing Co. and Kodansha.

[4] Gihman I.I., Skorohod A.V. (1982). *Stochastic differential equations and its applications*. Naukova Dumka. Kiev.