MINIMAX EXTRAPOLATION PROBLEM FOR PERIODICALLY CORRELATED PROCESSES

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Abstract

The problem of mean square optimal estimation of linear functional from periodically correlated stochastic process is considered. This problem is investigated under conditions of spectral certainty and spectral uncertainty.

1 Introduction

Methods of solution of the problem of estimation of the unknown values of stationary stochastic processes (extrapolation, interpolation and filtering problems) with known spectral densities of processes are developed by A.N. Kolmogorov [3], N. Wiener, A.M. Yaglom. In the case where complete information on the spectral densities is impossible, but a set of admissible spectral densities is given, the minimax-robust approach to estimation problem is reasonable. That is we find estimate that minimizes the mean square error for all spectral densities from a given class of densities simultaneously. M.P. Moklyachuk, O.Yu. Masyutka [4] studied the extrapolation, interpolation and filtering problems for stationary processes and sequences.

In the paper by E.G. Gladyshev [2] investigation of periodically correlated processes was started. Minimax estimation problems for linear functionals from periodically correlated sequences were studied in articles by I.I. Dubovetska, O.Yu. Masyutka, M.P. Moklyachuk in [1].

2 The problem statement

We consider the problem of mean square optimal linear estimation of the functional

$$A\zeta = \int_0^{\infty} a(t)\zeta(t)dt$$

that depends on the unknown values of the mean square continuous periodically correlated stochastic process $\zeta(t)$ based on observations of the process $\zeta(t)$ for $t < 0$. The function $a(t), t \in R_+$, satisfies the condition $\int_0^{\infty} |a(t)|dt < \infty$.

3 Periodically correlated processes and generated stationary sequences

Definition 1 *[2] Mean square continuous stochastic process* $\zeta : R \to H = L_2(\Omega, \mathcal{F}, P)$, $\mathsf{E}\zeta(t) = 0$, *is called periodically correlated (PC) with period* T, *if its correlation function*

$K(t+u,u) = \mathsf{E}\zeta(t+u)\overline{\zeta(u)}$ *for all* $t,u \in R$ *and some fixed* $T>0$ *is such that*

$$K(t+u,u) = \mathsf{E}\zeta(t+u+T)\overline{\zeta(u+T)} = K(t+u+T,u+T).$$

Let $\{\zeta(t), t \in R\}$ be a PC process. We construct the sequence of stochastic functions

$$\{\zeta_j(u) = \zeta(u+jT), u \in [0,T), j \in Z\}. \tag{1}$$

The sequence (1) forms the $L_2([0,T);H)$-valued stationary stochastic sequence $\{\zeta_j, j \in Z\}$. If we define in $L_2([0,T);R)$ the orthonormal basis $\{\widetilde{e}_k = \frac{1}{\sqrt{T}} e^{2\pi i \{(-1)^k \left[\frac{k}{2}\right]\} u/T}$, $k=1,2,\dots\}$, $\langle \widetilde{e}_j, \widetilde{e}_k \rangle = \delta_{kj}$, the generated vector stationary sequence $\{\zeta_j, j \in Z\}$ can be represented in the form

$$\zeta_j = \sum_{k=1}^{\infty} \zeta_{kj} \widetilde{e}_k, \quad \zeta_{kj} = \langle \zeta_j, \widetilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T \zeta_j(v) e^{-2\pi i \{(-1)^k \left[\frac{k}{2}\right]\} v/T} dv. \tag{2}$$

4 The classical method of extrapolation

Taking into account the decomposition (2) of generated stationary sequence $\{\zeta_j, j \in Z\}$, the functional $A\zeta$ can be represented in the form

$$A\zeta = \int_0^{\infty} a(t)\zeta(t)dt = \sum_{j=0}^{\infty} \int_0^T a_j(u)\zeta_j(u)du = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{kj}\zeta_{kj} = \sum_{j=0}^{\infty} \vec{a}_j^{\top} \vec{\zeta}_j,$$

$$\vec{\zeta}_j = (\zeta_{kj}, k \geq 1)^{\top}, \quad \vec{a}_j = (a_{kj}, k \geq 1)^{\top} = (a_{1j}, a_{3j}, a_{2j}, \dots, a_{2k+1,j}, a_{2k,j}, \dots)^{\top}.$$

The classical Kolmogorov projection method [3] allows us to find main characteristics of the optimal linear estimate of the functional $A\zeta$.

Theorem 1 *Let* $\{\zeta(t), t \in R\}$ *be a PC process such that the vector stationary sequence* $\{\zeta_j, j \in Z\}$ *generated by relation (1) has the spectral density* $f(\lambda)$ *that satisfies the minimality condition* $\int_{-\pi}^{\pi} Tr[(f(\lambda))^{-1}] d\lambda < \infty$.

The spectral characteristic $h(f)$ *and the mean square error* $\Delta(f)$ *of the optimal linear estimate of the functional* $A\zeta$ *from observations of the process* $\zeta(t)$ *for* $t<0$ *are given by formulas*

$$h^{\top}(f) = A^{\top}(e^{i\lambda}) - C^{\top}(e^{i\lambda})[f(\lambda)]^{-1}, \tag{3}$$

$$\Delta(f) = \langle \mathbf{c}, \mathbf{a} \rangle, \tag{4}$$

where $A(e^{i\lambda}) = \sum_{j=0}^{\infty} \vec{a}_j e^{ij\lambda}$, $C(e^{i\lambda}) = \sum_{j=0}^{\infty} \vec{c}_j e^{ij\lambda}$, $\mathbf{a} = \{\vec{a}_j\}_{j=0}^{\infty}$, $\mathbf{c} = \{\vec{c}_j\}_{j=0}^{\infty} = \mathbf{B}^{-1}\mathbf{a}$, $\mathbf{B} = \{B(l,j)\}_{l,j=0}^{\infty}$ *is matrix with elements* $B(l,j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(f(\lambda))^{-1}\right]^{\top} e^{i(j-l)\lambda} d\lambda$.

5 Least favorable spectral densities in the class D_0

Consider the minimax-robust approach to the problem of estimation of the functional $A\zeta$ under the condition that the spectral density $f(\lambda)$ of generated vector stationary sequence $\{\zeta_j, j \in Z\}$ constructed by relation (1) belongs to the class

$$D_0 = \left\{ f(\lambda) | \frac{1}{2\pi} \int_{-\pi}^{\pi} Tr\, f(\lambda) d\lambda = P_\zeta \right\}.$$

The spectral density $f(\lambda)$ of the regular stationary sequence $\{\zeta_j, j \in Z\}$ admits the canonical factorization

$$f(\lambda) = P(\lambda)P^*(\lambda), \quad P(\lambda) = \sum_{u=0}^{\infty} d(u)e^{-iu\lambda}, \quad d(u) = \{d_{km}(u)\}_{k=\overline{1,\infty}}^{m=\overline{1,M}}.$$

Applying the Lagrange multipliers method to the conditional extremum problem

$$\Delta(f) = \|\mathbf{Ad}\|^2 \to max, \quad f(\lambda) = P(\lambda)P^*(\lambda) \in D_0, \tag{5}$$

where $||\mathbf{Ad}||^2 = \sum_{l=0}^{\infty} ||(\mathbf{Ad})_l||^2$, $(\mathbf{Ad})_l = \sum_{j=l}^{\infty} \vec{a}_j^\top d(j-l), l \geq 0$, we can derive relations

$$\sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \overline{\vec{a}_{r+p}} \vec{a}_{s+p}^\top d(s) = \alpha^2 d(r), \quad r = 0, 1, \ldots, \tag{6}$$

$$\|\mathbf{d}\|^2 = \sum_{u=0}^{\infty} \|d(u)\|^2 = P_\zeta, \tag{7}$$

that determine solution $f^0(\lambda)$ of (5). The following theorem holds true.

Theorem 2 *The spectral density*

$$f^0(\lambda) = \left(\sum_{u=0}^{\infty} d^0(u)e^{-iu\lambda} \right) \left(\sum_{u=0}^{\infty} d^0(u)e^{-iu\lambda} \right)^*$$

of the moving average sequence $\{\zeta_j, j \in Z\}$ with components

$$\zeta_{kj} = \sum_{u=-\infty}^{j} \sum_{m=1}^{M} d_{km}^0(j-u)\varepsilon_m(u)$$

is the least favorable in the class D_0 for the optimal linear estimation of the functional $A\zeta$. The sequence of matrices $\mathbf{d^0} = \{d^0(u), u = 0, 1, \ldots\}$ is determined by relations (6), (7). The minimax spectral characteristic $h(f^0)$ is given by the formula

$$h^\top(f^0) = A^\top(e^{i\lambda}) - S^0(e^{i\lambda})Q^0(\lambda),$$

where $S^0(e^{i\lambda}) = \sum_{l=0}^{\infty} (\mathbf{Ad^0})_l e^{il\lambda}$, $Q^0(\lambda) = \{q_{mk}^0(\lambda)\}_{m=\overline{1,M}}^{k=\overline{1,\infty}}$ is the matrix function determined by equation $Q^0(\lambda)P^0(\lambda) = I_M$.

6 Least favorable spectral densities in the class D_0^-

Consider the minimax-robust approach to the problem of estimation of the functional $A\zeta$ under the condition that the spectral density $f(\lambda)$ belongs to the class

$$D_0^- = \left\{ f(\lambda) | \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\lambda) d\lambda = V \right\},$$

where $V = \{v_{kn}\}_{k,n=1}^{\infty}$ is a given nonnegative matrix with $detV \neq 0$. The least favorable in the class D_0^- spectral density gives solution to the problem

$$\Delta(h(f^0); f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (C^0(e^{i\lambda}))^{\top} (f^0(\lambda))^{-1} f(\lambda) (f^0(\lambda))^{-1} \overline{C^0(e^{i\lambda})} d\lambda \to sup, \;\; f \in D_0$$

and is determined by relation

$$\left[(f^0(\lambda))^{-1} \right]^{\top} C^0(e^{i\lambda}) = \left[(f^0(\lambda))^{-1} \right]^{\top} \vec{\alpha}, \tag{8}$$

where $\vec{\alpha}$ is the Lagrange multiplier. Let denote by $V^{\top}(l-j) = B^0(l,j)$, $(\vec{a}_0)^{-1}$ define from the equality $((\vec{a}_0)^{-1})^{\top} \vec{a}_0 = 1$. Then the following theorem holds true.

Theorem 3 *Suppose the coefficients $\{\vec{a}_j, j = 0, 1, \dots\}$, that determine the functional $A\zeta$, are such that the matrix function $\sum_{u=-\infty}^{\infty} V(u) e^{iu\lambda}$, where*

$$V(u) = V^*(-u) = V(\vec{a}_0)^{-1} \vec{a}_u^{\top}, \quad u = 0, 1, \dots,$$

is positive definite and has nonzero determinant. Then the spectral density

$$f^0(\lambda) = \left(\sum_{u=-\infty}^{\infty} V(u) e^{iu\lambda} \right)^{-1}$$

is the least favorable in the class D_0^- for the optimal linear estimation of the functional $A\zeta$. The minimax spectral characteristic $h(f^0)$ is given by the formula

$$h(f^0) = -\sum_{u=1}^{\infty} \overline{V(u)} (V^{\top})^{-1} \vec{a}_0 e^{-iu\lambda.}$$

References

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