APPLICATION OF RESIDUAL EMPIRICAL PROCESSES TO ROBUST LINEAR HYPOTHESES TESTING IN AUTOREGRESSION

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Abstract

The paper deals with the property of asymptotic uniform linearity of residual empirical processes for AR(p) when observations contain outliers. We apply the result to construct robust GM–tests for linear hypotheses. The scheme of data contamination by additive single outliers with the intensity $O(n^{-1/2})$, n is data level, is considered.

1 Introduction. Formulation of the problem

In this paper we construct nonparametric robust generalized M-test (GM-test) for linear hypotheses in autoregressive AR(p) model. Consider the model

$$u_t = \beta_1 u_{t-1} + \cdots + \beta_p u_{t-p} + \varepsilon_t, \quad t \in \mathbb{Z}. \tag{1.1}$$

Here $\{\varepsilon_t\}$ — i.i.d. random variables with unknown distribution function G and density g, $E\varepsilon_1 = 0$, $E\varepsilon_1^2 < \infty$; $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$ — vector of unknown parameters. We suppose the coefficients $\beta_1, \ldots, \beta_p$ are such that the roots of the characteristic equation $x^p = \beta_1 x^{p-1} + \cdots + \beta_p$ are less than one in absolute value. This condition guarantees that (1.1) has an a.s. unique stationary solution. Let the observations contain outliers and be of the following form

$$y_t = u_t + z_t^{\gamma_n} \xi_t, \quad t = 0, 1, \ldots, n. \tag{1.2}$$

In (1.2) $\{u_t\}$ is the sample from (1.1); $\{z_t^{\gamma_n}\}$ — i.i.d. random variables with Bernoulli distribution $\mathrm{Br}\,(\gamma_n)$, $\gamma_n = \min(1, n^{-1/2}\gamma)$, parameter $\gamma \geq 0$ is unknown; $\{\xi_t\}$ — i.i.d. random variables with an unknown distribution μ; the sequences $\{u_t\}$, $\{z_t^{\gamma_n}\}$, $\{\xi_t\}$ are independent to each other. Sequence $\{\xi_t\}$ is interpreted as a sequence of outliers (contamination), γ_n is a contamination level. Scheme (1.2) is the local variant of contamination scheme from [6].

Let's represent $\boldsymbol{\beta}$ as $\boldsymbol{\beta}^T = (\boldsymbol{\beta}^{(1)^T}, \boldsymbol{\beta}^{(2)^T})$, where the vectors $\boldsymbol{\beta}^{(i)}$, $i = 1, 2$, have the dimensions m and $p - m$ respectively, $1 \leq m < p$. The linear hypothesis has the form $H_0\colon \boldsymbol{\beta}^{(2)} = \boldsymbol{\beta}_0^{(2)}$. Here $\boldsymbol{\beta}_0^{(2)}$ is a known vector and $\boldsymbol{\beta}^{(1)}$ is an interfering parameter. Introduce the local alternatives $H_{1n}(\boldsymbol{\tau})\colon\ \boldsymbol{\beta} = \boldsymbol{\beta}_n := \boldsymbol{\beta}_0 + n^{-1/2}\boldsymbol{\tau}$, where $\boldsymbol{\beta}_0^T = (\boldsymbol{\beta}^{(1)^T}, \boldsymbol{\beta}_0^{(2)^T})$, $\boldsymbol{\tau}^T = (\boldsymbol{\tau}^{(1)^T}, \boldsymbol{\tau}^{(2)^T}) \in \mathbb{R}^p$ is a constant vector with subvectors of dimensions m and $p - m$ respectively. Thus, the unknown parameter $\boldsymbol{\beta}^{(1)}$ is admitted to variate with a magnitude of order $O(n^{-1/2})$ in the alternatives $H_{1n}(\boldsymbol{\tau})$.

In order to verify H_0 in case of absence of outliers least squares tests, rank, sign, and M-tests are often used. One of the general aims of this paper is to construct GM-test which would be robust against outliers $\{\xi_t\}$ using $\{y_t\}$.

Usually GM-tests are constructed from GM-estimators. Let's describe this method. Consider model (1.1) without outliers. For some functions φ, ψ and arbitrary $\boldsymbol{\alpha} \in \mathbb{R}^p$ introduce vector

$$\boldsymbol{L}_n(\boldsymbol{\alpha}) := \big(L_{n1}(\boldsymbol{\alpha}), L_{n2}(\boldsymbol{\alpha}), \ldots, L_{np}(\boldsymbol{\alpha})\big)^T,$$
$$L_{nj}(\boldsymbol{\alpha}) := n^{-1/2} \sum_{t=1}^{n} \varphi(u_{t-j})\psi(u_t - \alpha_1 u_{t-1} - \cdots - \alpha_p u_{t-p}).$$

GM-estimator is defined as $\sqrt{n}$-consistent solution of the equation $\boldsymbol{L}_n(\boldsymbol{\alpha}) = \mathbf{0}$. Denote the solution as $\widehat{\boldsymbol{\beta}}_{n,GM}$. For φ and ψ satisfying definite conditions estimator $\widehat{\boldsymbol{\beta}}_{n,GM}$ is asymptotically Gaussian (see details in [3]). Using this property last $p-m$ coordinates of $\widehat{\boldsymbol{\beta}}_{n,GM}$ can be used for verifying H_0 in standard way.

Consider statistic $\boldsymbol{L}_n^Y(\boldsymbol{\alpha})$ which is constructed from $\{y_t\}$ in the same way as $\boldsymbol{L}_n(\boldsymbol{\alpha})$ is constructed from $\{u_t\}$. In this paper tests are based not on GM-estimators but on statistic $\boldsymbol{L}_n^Y(\widehat{\boldsymbol{\beta}}_n)$ transformed in a special way. Here $\widehat{\boldsymbol{\beta}}_n$ is an arbitrary $\sqrt{n}$-consistent estimator of $\boldsymbol{\beta}$. Note that for tests of this kind functions φ and ψ should satisfy weaker regularity conditions than for tests based on GM-estimators. In case of $\varphi(x) = x$ our test is asymptotically equivalent to M-test constructed in [5].

Asymptotic properties of constructed test are considered in the alternatives $H_{1n}(\boldsymbol{\tau})$ using representation of $\boldsymbol{L}_n^Y(\boldsymbol{\alpha})$ as an integral functional of residual weighted empirical process. We obtain asymptotic uniform linearity (AUL) of these processes and hence $\boldsymbol{L}_n^Y(\boldsymbol{\alpha})$. In general the property of AUL of special-type residual processes has been widely used for parameter estimation and GM–estimation particularly in different models without contamination. For example nonlinear models with additive noises were considered in [4], ARCH models — in [2].

Denote the power of constructed test in alternatives $H_{1n}(\boldsymbol{\tau})$ as $W_n(\boldsymbol{\tau}, \gamma, \mu)$. In this work we obtain its limit power $W(\boldsymbol{\tau}, \gamma, \mu) := \lim_{n\to\infty} W_n(\boldsymbol{\tau}, \gamma, \mu)$. Let $W(\boldsymbol{\tau}) := W(\boldsymbol{\tau}, 0, \mu)$ be the limit power of the test in scheme (1.1) without outliers. It turns out that for small values of γ limit powers $W(\boldsymbol{\tau}, \gamma, \mu)$ and $W(\boldsymbol{\tau})$ are closed uniformly with respect to $\mu \in \mathfrak{M}_2$ (see statement (2.5)). Here $\mathfrak{M}_2$ is a class of outliers with finite second moment. Expression (2.5) means that the test has qualitatively robust limit power. This definition of robustness against outliers had been used before. For example in [1] a sign test for linear hypotheses in AR(p) was constructed and corresponding relation of the type of (2.5) for this test was proved.

2 Main results

Consider the scheme of the type (1.2). Using observations $\{y_t\}$ construct the vector $\boldsymbol{Y}_{t-1} := (y_{t-1}, y_{t-2}, \ldots, y_{t-p})^T$, $t = 1, \ldots, p$. For the sake of brevity denote vector function $\boldsymbol{\varphi}(x_1, x_2, \ldots, x_p) := (\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_p))^T$. Define weighted residual empirical process as $\boldsymbol{v}_n(\boldsymbol{\alpha}, x, \gamma, \mu) := n^{-1/2} \sum_{t=1}^n \boldsymbol{\varphi}(\boldsymbol{Y}_{t-1}) I(y_t - \boldsymbol{\alpha}^T \boldsymbol{Y}_{t-1} \leq x)$, $I(\cdot)$ is an indicator function. Then the integral functional is constructed as

$$\boldsymbol{L}_n^Y(\boldsymbol{\alpha}) := \int_{-\infty}^{\infty} \psi(x) d\boldsymbol{v}_n(\boldsymbol{\alpha}, x, \gamma, \mu). \tag{2.1}$$

Now define σ-algebras $\mathcal{F}_{t-1} := \sigma\{\varepsilon_i, i \leq t-1; (\xi_j, z_j^{\gamma_n}), 0 \leq j \leq t\},\ t = 1, \ldots, n$. Denote $\eta_t(\boldsymbol{\beta}, \gamma) := z_t^{\gamma_n}\xi_t - \beta_1 z_{t-1}^{\gamma_n}\xi_{t-1} - \beta_2 z_{t-2}^{\gamma_n}\xi_{t-2} - \cdots - \beta_p z_{t-p}^{\gamma_n}\xi_{t-p}$ and introduce empirical process which is conditionally centered with respect to $\mathcal{F}_{t-1}$

$$\begin{aligned} \boldsymbol{u_n}(\boldsymbol{\theta}, x, \gamma, \mu) := n^{-1/2} \sum_{t=1}^n \boldsymbol{\varphi}(\boldsymbol{Y}_{t-1}) \Big[& I\big(\varepsilon_t \leq x + n^{-1/2}\boldsymbol{\theta}^T \mathbf{Y_{t-1}} - \eta_t(\boldsymbol{\beta}, \gamma)\big) \\ & - G\big(x + n^{-1/2}\boldsymbol{\theta}^T \mathbf{Y_{t-1}} - \eta_t(\boldsymbol{\beta}, \gamma)\big)\Big]. \end{aligned} \tag{2.2}$$

To formulate the property of AUL when $\gamma \neq 0$ we need following conditions to hold:
Condition (i). $\sup_x |\varphi(x)| < \infty$.
Condition (ii). G is twice differentiable, $G' = g$, $\sup_x |g'(x)| < \infty$.
Condition (iii). $\mathbf{E}|\xi_1|^2 < \infty$.

Theorem 1. *Let the conditions (i)–(iii) hold. Let $H_{1n}(\boldsymbol{\tau})$ be valid. Then*

$$\sup_{|\boldsymbol{\theta}| \leq \Theta,\ x \in \mathbb{R}^1} |\boldsymbol{u}_n(\boldsymbol{\theta}, x, \gamma, \mu) - \boldsymbol{u}_n(\mathbf{0}, x, \gamma, \mu)| \xrightarrow{P} 0, \quad n \to \infty, \quad 0 \leq \Theta < \infty$$

Let $u_{1-p}^0, \ldots, u_n^0$ be a sample from the strictly stationary solution to equation (1.1) for $\boldsymbol{\beta} = \boldsymbol{\beta_0}$. Denote

$$\begin{aligned} \boldsymbol{\Delta}(\mu) &:= (\Delta_1(\mu), \Delta_2(\mu), \ldots, \Delta_p(\mu))^T, \\ \Delta_j(\mu) &:= E\varphi(u_{2-j}^0 + \xi_{2-j})\psi(\varepsilon_2 - \beta_{0j}\xi_{2-j}) + E\varphi(u_{2-j}^0)E\psi(\varepsilon_1 + \xi_1) \\ &\quad + \sum_{i=1,\ i \neq j}^p E\varphi(u_{2-j}^0)E\psi(\varepsilon_2 - \beta_{0i}\xi_{2-i}), \end{aligned}$$

$\boldsymbol{\Delta}(\mu)$ characterizes the outliers influence on $\boldsymbol{L}_n^Y(\boldsymbol{\alpha})$ and test statistic.

Define the following condition for the function ψ.
Condition (iv) Variation $\mathrm{Var}\,|_{-\infty}^{\infty}\,[\psi] < \infty$, $E\psi(\varepsilon_1) = 0$, $\int_{-\infty}^{\infty} g(x) d\psi(x) \neq 0$.

Using $\{u_t^0\}$ construct vectors $\boldsymbol{U}_{t-1}^0 := (u_{t-1}^0, \ldots, u_{t-p}^0)^T$, $t = 1, \ldots, n$. Let $\widetilde{\boldsymbol{L}}_n(\boldsymbol{\beta}_0) := n^{-1/2} \sum_{t=1}^n \boldsymbol{\varphi}(\boldsymbol{U}_{t-1}^0)\psi(\varepsilon_t)$. Define matrix

$$\mathbf{C} := \int_{-\infty}^{\infty} g(x) d\psi(x)\, \mathbf{E}\boldsymbol{\varphi}(\boldsymbol{U}_0^0)(\boldsymbol{U}_0^0)^T.$$

Under theorem 1 and relation (2.1) we obtain

Theorem 2. *Let the conditions (i)–(iv) hold. Let $\varphi(x)$ be continuous a.s. If $H_{1n}(\boldsymbol{\tau})$ is valid then for any $0 \leq \Theta < \infty$ the following uniform convergence holds*

$$\sup_{|\boldsymbol{\theta}| \leq \Theta} \big| \boldsymbol{L}_n^Y(\boldsymbol{\beta}_n + n^{-1/2}\boldsymbol{\theta}) - \widetilde{\boldsymbol{L}}_n(\boldsymbol{\beta}_0) + \mathbf{C}\boldsymbol{\theta} - \gamma\boldsymbol{\Delta}(\mu) \big| \xrightarrow{P} 0, \quad n \to \infty. \tag{2.3}$$

Now proceed to robust GM-test construction. Denote $\widehat{\boldsymbol{\beta}}_{n0}^T = (\widehat{\boldsymbol{\beta}}_{n0}^{(1)^T}, \boldsymbol{\beta}_0^{(2)^T})$, where $\widehat{\boldsymbol{\beta}}_{n0}^{(1)}$ is an arbitrary $\sqrt{n}$–consistent estimator of interfering parameter $\boldsymbol{\beta}^{(1)}$. Denote consistent estimator (in $H_{1n}(\boldsymbol{\tau})$) of matrix $\mathbf{C}$ as $\widehat{\mathbf{C}}_n$. Let π be the orthogonal projection onto last $(p-m)$ components. Let $\det \mathbf{C} \neq 0$. Take

$$\Lambda_{n,Y}^{\pi} := \left[\pi \circ \widehat{\mathbf{C}}_n^{-1} \boldsymbol{L}_n^Y(\widehat{\boldsymbol{\beta}}_{n0})\right]^T \widehat{\mathbf{J}}_n^{-1} \left[\pi \circ \widehat{\mathbf{C}}_n^{-1} \boldsymbol{L}_n^Y(\widehat{\boldsymbol{\beta}}_{n0})\right] \tag{2.4}$$

as a statistic of the test. Here $\widehat{\mathbf{J}}_n$ is an arbitrary consistent estimator of $\mathbf{J}$ which is a covariance matrix of $\pi \circ \widehat{\mathbf{C}}_n^{-1} \boldsymbol{L}_n^Y(\widehat{\boldsymbol{\beta}}_{n0})$. Denote $\boldsymbol{a}(\gamma, \mu) := \gamma \mathbf{C}^{-1} \boldsymbol{\Delta}(\mu)$, $\boldsymbol{a}^T(\gamma, \mu) = (\boldsymbol{a}^{(1)T}, \boldsymbol{a}^{(2)T})$. With the help of expansion (2.3) one can prove the following

Theorem 3. *Let the conditions of the Theorem 2 hold. Then for $n \to \infty$*

$$\Lambda_{n,Y}^{\pi} \xrightarrow{d} \chi^2(p-m, \lambda^2), \quad \lambda^2 = |\mathbf{J}^{-1/2}(\boldsymbol{\tau}^{(2)} + \boldsymbol{a}^{(2)})|^2$$

In the alternatives $H_{1n}(\boldsymbol{\tau})$ the power of the test based on $\Lambda_{n,Y}^{\pi}$ is $W_n(\boldsymbol{\tau}, \gamma, \mu) = P_{\boldsymbol{\beta}_n}(\Lambda_{n,Y}^{\pi} > \chi_{1-\alpha}^{p-m})$, $\chi_{1-\alpha}^{p-m}$ is $(1-\alpha)$–quantile of the distribution of $\chi^2(p-m)$. Due to the theorem 3 $\lim_{n\to\infty} W_n(\boldsymbol{\tau}, \gamma, \mu) = W(\boldsymbol{\tau}, \gamma, \mu) = 1 - F_{p-m}(\chi_{1-\alpha}^{p-m}, \lambda^2)$, and the test has the asymptotic confidence level α. Let $W_n(\boldsymbol{\tau})$ be the power of test in scheme (1.1) without outliers. Then due to the theorem 3 there exists $\lim_{n\to\infty} W_n(\boldsymbol{\tau}) = W(\boldsymbol{\tau}) := W(\boldsymbol{\tau}, 0, \mu)$. The following theorem characterizes the robustness of GM–test (2.4) against outliers or, if to be more precise, the qualitative robustness of its limit power (see [1]).

Theorem 4. *Let the conditions of the Theorem 2 hold, then*

$$\sup_{\mu \in \mathfrak{M}_2} |W(\boldsymbol{\tau}, \gamma, \mu) - W(\boldsymbol{\tau})| \to 0, \quad \gamma \to 0. \tag{2.5}$$

Thus the family of limit powers $\{W(\boldsymbol{\tau}, \gamma, \mu)\}_\mu$ is equicontinuous in γ at $\gamma = 0$.

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