PARAMETER ESTIMATION IN THE MODELS WITH LONG-RANGE DEPENDENCE

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Abstract

We consider a problem of statistical estimation of an unknown drift parameter for a stochastic differential equation driven by fractional Brownian motion. Two estimators based on discrete observations of solution to the stochastic differential equations are constructed. It is proved that the estimators converge almost surely to the parameter value, as the observation interval expands and the interval between observations vanishes. A bound for the rate of convergence is given. As an auxiliary result of independent interest we establish global estimates for fractional derivative of fractional Brownian motion.

1 Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ with the covariance $\mathsf{E}\left[\,B_t^H B_s^H\,\right] = \frac{1}{2}(s^{2H}+t^{2H}-|t-s|^{2H})$. Stochastic differential equations driven by fBm has been an active research are for the last two decades. Main reason is that such equations seem to be one of the most suitable tools to model long-range dependence in many applied areas, such as physics, finance, biology, network studies etc.

This paper deals with statistical estimation of drift parameter for a stochastic differential equation with fBm by discrete observation of its solution. We propose two new estimators and prove their strong consistency under the so-called "high-frequency data" assumption that the horizon of observations tends to infinity, while the interval between them goes to zero. Moreover, we obtain almost sure upper bounds for the rate of convergence of the estimators. The estimators proposed go far away from being maximum likelihood estimators, and this is their crucial advantage, because they keep strong consistency but they are not complicated technically and are convenient for the simulations.

2 Preliminaries

2.1 Fractional Brownian motion

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with the covariance $\mathsf{E}\left[\,B_t^H B_s^H\,\right] = \frac{1}{2}(s^{2H}+t^{2H}-|t-s|^{2H})$. It is well known that B^H has a modification with almost surely continuous paths (even Hölder continuous of any order up to H), and further we will assume that it is continuous itself.

In what follows we assume that the Hurst parameter $H \in (1/2, 1)$ is fixed. In this case, the integral with respect to the fBm B^H will be understood in the generalized Lebesgue–Stieltjes sense (see [4]). Its construction uses the fractional derivatives, defined for $a < b$ and $\alpha \in (0,1)$ as

$$\big(D_{a+}^{\alpha} f\big)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}} du \right),$$
$$\big(D_{b-}^{1-\alpha} g\big)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}} du \right).$$

Provided that $D_{a+}^{\alpha} f \in L_1[a,b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_\infty[a,b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue-Stieltjes integral $\int_a^b f(x) dg(x)$ is defined as

$$\int_a^b f(x) dg(x) = e^{i\pi\alpha} \int_a^b \big(D_{a+}^{\alpha} f\big)(x) \big(D_{b-}^{1-\alpha} g_{b-}\big)(x) dx. \tag{1}$$

It follows from Hölder continuity of B^H that for $\alpha \in (1-H, 1)$ $D_{b-}^{1-\alpha} B_{b-}^H \in L_\infty[a,b]$ a.s. Then for a function f with $D_{a+}^{\alpha} f \in L_1[a,b]$ we can define integral with respect to B^H through (1):

$$\int_a^b f(x)\, dB^H(x) := e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x) (D_{b-}^{1-\alpha} B_{b-}^H)(x)\, dx. \tag{2}$$

2.2 Estimate of derivative of fractional Brownian motion

In order to estimate integrals with respect to fractional Brownian motion, we need to estimate the fractional derivative of B^H. Let some $\alpha \in (1-H, 1/2)$ be fixed until the rest of this paper. Denote for $t_1 < t_2$

$$Z(t_1, t_2) = \left(D_{t_2-}^{1-\alpha} B_{t_2-}^H\right)(t_1) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{B_{t_1}^H - B_{t_2}^H}{(t_2-t_1)^{1-\alpha}} + (1-\alpha) \int_{t_1}^{t_2} \frac{B_{t_1}^H - B_u^H}{(u-t_1)^{2-\alpha}} du \right).$$

The following proposition is a generalization of [1, Theorem 3].

Theorem 1. *For any $\gamma > 1/2$,*

$$\xi_{H,\alpha,\gamma} := \sup_{0 \le t_1 < t_2 \le t_1+1} \frac{|Z(t_1,t_2)|}{(t_2-t_1)^{H+\alpha-1} \left(|\log(t_2-t_1)|^{1/2} + 1 \right) (\log(t_2+3))^{\gamma}} \tag{3}$$

is finite almost surely.

Moreover, there exists $c_{H,\alpha,\gamma} > 0$ such that $\mathsf{E}\left[\exp\left\{x \xi_{H,\alpha,\gamma}^2\right\}\right] < \infty$ for $x < c_{H,\alpha,\gamma}$.

2.3 Estimates for solution of SDE driven by fractional Brownian motion

Consider a stochastic differential equation

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^H, \tag{4}$$

where X_0 is a non-random coefficient. In [2], it is shown that this equation has a unique solution under the following assumptions: there exist constants $\delta \in (1/H-1, 1]$, $K > 0$, $L > 0$ and for every $N > 0$ there exists $R_N > 0$ such that

(A) $|a(x)| + |b(x)| \le K \quad$ for all $x, y \in \mathbb{R}$,

(B) $|a(x) - a(y)| + |b(x) - b(y)| \le L\,|x - y| \quad$ for all $x, y \in \mathbb{R}$,

(C) $|b'(x) - b'(y)| \le R_N\,|x - y|^\delta \quad$ for all $x \in [-N, N], y \in [-N, N]$.

Fix some $\beta \in (1/2, H)$. Denote for $t_1 < t_2$

$$\Lambda_\beta(t_1, t_2) = 1 \vee \sup_{t_1 \le u < v \le t_2} \frac{|Z(u, v)|}{(v-u)^{\beta+\alpha-1}}.$$

Theorem 2. *There exists a constant $M_{\alpha,\beta}$ depending on α, β, K, and L such that for any $t_1 \ge 0, t_2 \in (t_1, t_1 + 1]$*

$$|X_{t_2} - X_{t_1}| \le M_{\alpha,\beta}\left(\Lambda_\beta(t_1, t_2)(t_2 - t_1)^\beta + \Lambda_\beta(t_1, t_2)^{1/\beta}(t_2 - t_1)\right).$$

Corollary 1. *For any $\gamma > 1/2$, there exist random variables ξ and ζ such that for all $t_1 \ge 0$, $t_2 \in (t_1, t_1 + 1]$*

$$|X_{t_2} - X_{t_1}| \le \zeta(t_2 - t_1)^\beta\,(\log(t_2 + 3))^\kappa, \quad \Lambda_\beta(t_1, t_2) \le \xi(\log(t_2 + 3))^{\kappa\beta},$$

where $\kappa = \gamma/\beta$. Moreover, there exists some $c > 0$ such that $\mathsf{E}\left[\exp\{x\xi^2\}\right] < \infty$ and $\mathsf{E}\left[\exp\left\{x\zeta^{2\beta}\right\}\right] < \infty$ for $x < c$. In particular, all moments of ξ and ζ are finite.

3 Drift parameter estimation

Now we turn to problem of drift parameter estimation in equations of type (4). Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to C[0, \infty)$ be a stochastic process. Consider a family of probability measures $\left\{\mathsf{P}^\theta, \theta \in \mathbb{R}\right\}$ on $(\Omega, \mathcal{F})$ such that for each $\theta \in \mathbb{R}$, $\mathcal{F}$ is P^θ-complete, and there is an fBm $\left\{B_t^{H,\theta}, t \ge 0\right\}$ on $(\Omega, \mathcal{F}, \mathsf{P}^\theta)$ such that X solves a parametrised version of (4)

$$X_t = X_0 + \theta\int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^{H,\theta}. \tag{5}$$

Our main problem is the following: to construct an estimator for θ based on discrete observations of X. Specifically, we will assume that for some $n \geq 1$ we observe values $X_{t_n^k}$ at the following uniform partition of $[0, 2^n]$: $t_k^n = k2^{-n}$, $k = 0, 1, \ldots, 2^{2n}$.

Fix the parameters $\alpha \in (1-H, 1/2)$, $\beta \in (1-\alpha, H)$, $\gamma > 1/2$ and $\kappa = \gamma/\beta$.

In order to proceed, we need another technical assumption, in addition to conditions (A)–(C):

(D) there exist a constant $M > 0$ such that for all $x \in \mathbb{R}$

$$|a(x)| \geq M, \qquad |b(x)| \geq M.$$

Define an estimator

$$\hat{\theta}_n^{(1)} = \frac{\sum_{k=1}^{2^{2n}} (t_k^n)^{\lambda} (2^n - t_k^n)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\sum_{k=1}^{2^{2n}} (t_k^n)^{\lambda} (2^n - t_k^n)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right) \frac{1}{2^n}},$$

where $\lambda = 1/2 - H$.

Theorem 3. *With probability one, $\hat{\theta}_n^{(1)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η with all finite moments such that $\left|\hat{\theta}_n^{(1)} - \theta\right| \leq \eta n^{\kappa+\gamma} 2^{-\rho n}$, where $\rho = (1-H) \wedge (2\beta - 1)$.*

Consider a simpler estimator:

$$\hat{\theta}_n^{(2)} = \frac{\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\frac{1}{2^n}\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right)}.$$

This is a discretized maximum likelihood estimator for θ in equation (4), where B^H is replaced by Wiener process. Nevertheless, this estimator is consistent as well. Namely, we have the following result.

Theorem 4. *With probability one, $\hat{\theta}_n^{(2)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η' with all finite moments such that $\left|\theta_n^{(2)} - \theta\right| \leq \eta' n^{\kappa+\gamma} 2^{-\rho n}$.*

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