# Image Restoration Spectral Techniques

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***Abstract:*** *Since the single-channel framework of image restoration possesses serious conceptual and numerical problems, in this paper we present a general concept of image signal recovering. We also propose a space-variant restoration method using sliding spectral transforms. To provide image processing in real time, fast recursive algorithm for computing the sliding sinusoidal transform is utilized. Computer simulation results using a real image are provided and discussed.*

***Keywords***: Image restoration, spectral analysis, discrete cosine transform.

## 1. INTRODUCTION

In many applications observed images are often degraded owing to atmospheric turbulence, relative motion between a scene and a camera, nonuniform illumination, wrong focus, etc. Many different restoration techniques (linear, nonlinear, iterative, noniterative, deterministic, stochastic, etc.) optimized with respect to various criteria have been introduced [1]-[8]. The amount of *a priori* information about degradation, i.e., the size or shape of blurs, and the noise level, determines how mathematically ill-posed the problem is. The blind and nonblind deconvolutions have been extensively studied and many techniques have been proposed for their solution [3], [4]. They usually involve some regularization which assures various statistical properties of the image or constrains the estimated image and restoration filter according to some assumptions. This regularization is required to guarantee a unique solution and stability against noise and some model discrepancies. One of the most popular fundamental techniques is a linear minimum mean square error (LMMSE) method. It finds the linear estimate of the ideal image for which the mean square error between the estimate and the ideal image is minimum. The linear operator acting on the observed image to determine the estimate is obtained on the basis of *a priori* second order statistical information about the image and noise processes. In the case of stationary processes and space-invariant blurs, the LMMSE estimator takes the form of the Wiener filter. A Kalman filter determines the causal LMMSE estimate recursively. It is based on a state-space representation of the imaging system, and image data are used to define the state vectors. For images with sharp changes of intensity, the appropriate regularization is based on variational integrals. Minimization of the variational integrals preserves edges and fine details in the image and it was applied to blind restoration [6]–[8].

The objective of this presentation is twofold: we propose a general concept of image restoration based on an appropriate regularization of variational integrals, and then a fast space-variant restoration using sliding discrete transform coefficients is suggested. A sliding transform is based on the concept of short-time signal processing [9]-[10].

## 2. GENERAL APPROACH TO IMAGE RESTORATION

The image restoration problem is usually formulated as follows. Undistorted (original) image $z(\zeta,\eta)$ is recovered from the given equation:

$$Az + n = q(x,y) + n(x,y) = v(x,y), \quad , \tag{1}$$

where $A:\mathbb{Z}\to\mathbb{Q}$ ($\mathbb{Z},\mathbb{Q}$ are metric spaces) is a linear or nonlinear operator, $z\in\mathbb{Z}$, $q\in\mathbb{Q}$, $n(x,y)$ is a noise, $v(x,y)$ is an observed distorted image. A general approach for image restoration can be formulated using statistical estimation methods and the theory of solving of ill-posed problems [11]. The restoration problem is a typical inverse problem of mathematical physics, and, therefore, it can be correctly solved on the base of mathematical methods.

Assume that a method of image restoration to be considered is matched to the basic equation (1). So a general formulation of the restoration problem can be reduced to the following functional minimization:

$$z^* = \inf_{z\in Z} \rho_Q(Az,v), \tag{2}$$

where $\rho_Q$ is a certain metric in $\mathbb{Q}$. Note that various definitions of a distance $\rho_Q$ between two images may be used. It is easy to show that the solution of the optimization problem (2) is not unique even when the operator *A* and the distorted image *q*(*x*,*y*) are given exactly, without noise. We should use *a priori* information about the original image to obtain the unique and stable solution from the set of solutions. The simplest way to guarantee uniqueness and stability of the solution is to describe *a priory* information about the original image by means of a functional $\Omega(z)$ that possesses stabilizing properties [11]. In this case the image restoration problem can be reduced to conditional or unconditional optimization problem, in particular to the Tikhonov's minimization:

$$z^* = \inf_{z\in Z}\{\rho_Q(Az,v) + \alpha\Omega(z)\}, \tag{3}$$

where $\alpha$ is a parameter of regularization. Note that the statistical methods used in image restoration lead to optimization problems, which are similar to (3). For instance, using Bayes' strategy or MAP test we obtain the optimal estimation in the following form:

$$z^* = \inf_{z\in Z}\{-\ln p_2(Az-v) - \ln p_1(z)\}, \tag{4}$$

where $p_1(z)$ and $p_2(\xi)$ are a priori probability densities of the original image $z(\zeta,\eta)$ and additive noise $n(x,y) = Az - v$. The main difference between the regularization method of image restoration in (3) and the statistical method (4) is the existence of the regularization parameter $\alpha$ in (3). This leads to a family of solutions as a function of the parameter $\alpha$ . This allows us to control the visual quality of image restoration interactively in the absence of a mathematical criterion of visual image quality. If the space $\mathbb{Q}$ in (3) is defined as the Euclidian space with respect to the norm (*q*, *Bq*), where *B* is a positive defined operator, we obtain,

$$z^* = \inf_{z \in Z} \{ \|Az - v\|_B^2 + \alpha \Omega(z) \} . \quad (5)$$

Usually it is assumed that the original image is a smooth function with respect to the Sobolev space, and a stabilization functional in (5) is $\Omega(z) = \|z\|^q_{W^p_q}$ . Quadratic forms can be used in order to avoid nonlinear restoration algorithms. In particular, the usage of the Gaussian image model leads to the minimization of the quadratic form. In a discrete case it corresponds to the Sobolev norm for $p = 2$ in (5). On the other hand, the use of quadratic forms in image restoration brings undesirable results because of real images are not Gaussian.

Now suppose that images to be restored are functions of bounded variations. Therefore, it may be written as

$$z^* = \inf_{z \in Z} \{ \rho_Q (Az, v) + \alpha Var(z) \} , \quad (6)$$

It can be proved that in one-dimensional case when we hold fixed one of the variables, an image $f(x)$, $x \in [a,b]$ is a function of bounded variation where variation is

$$\overset{b}{\underset{a}{V}}(f) = \sup_{x_1 \ldots x_n} \sum_{k=2}^{n} | f(x_k) - f(x_{k-1}) | \quad . \quad (7)$$

It can be also shown, that the image $z(x, y)$ , $(x, y) \in D$ is a function of bounded variation also in the two-dimension case. It can be done for common definitions of multidimensional variations such as Arzela, Vitali, Tonelly and some other variations [12]. A different approach was proposed by Kronrod who introduced two functionals to characterize an image function of two variables. The functionals are given as follows [12]:

$$d_1(z) = \int_{-\infty}^{\infty} m_0(e_t) dt \, , d_2(z) = \int_{-\infty}^{\infty} m_1(e_t) dt \, , \quad (8)$$

where a set $e_t$ is a $t$ - level of the function $z(x, y)$, i.e. a set of points $(x, y)$ with values equal to $t$ , $m_0(e_t)$ - is the number of components of the set $e_t$ , $m_1(e_t)$ is the length of the set $e_t$ . The class of functions of bounded variations (8) is very extensive. In spite of this, a function of such class possesses a lot of good properties: they are differentiable almost everywhere, their Fourier series are convergent almost everywhere, etc. Note that numerous attempts to create a mathematical image model with the help of one functional were unsatisfactory. It can be done on the base of the two (independent in a certain way) functional. It is interesting to point out, that the first variation in (8), is not metric but a topological characteristic of an image. If the original image is a continuous differentiable function, the second variation can be represented as

$$d_2(z) = \int_a^b \int_c^g |grad\, z(x, y)| \, dx\, dy \, . \quad (9)$$

If the second variation is used, the restoration can be carried out as follows:

$$z^* = \inf_{z \in Z} \{ |Az - v|_B^2 + \alpha \int_a^b \int_c^g |grad\, z(x, y)| \, dx\, dy \} . \quad (10)$$

It is of interest to note that a nonlinear method of image restoration [13] based on anisotropic diffusion minimizes a functional that is identical to the Kronrod’s second variation.

Image restoration by using a functional minimization is a very complicated problem. The problem can be simplified by linearization of the used functionals. In this case, the Fourier-based techniques are applicable [14]. However, since the degraded image is defined on a limited area, it does not permit to apply the Fourier transform directly. To overcome the problem, additional procedures are required to extend the definition of degraded images [15]. In section 3, the restoration problem in (5) is simplified by image recovering in a sliding window. It is assumed that the signal is approximately stationary over the window area.

## 3. IMAGE RESTORATION WITH SLIDING TRANSFORMS

In this section we carry out the space-variant restoration using a sliding discrete cosine transform (DCT) coefficients. The sliding DCT is based on the concept of short-time signal processing [8]. The short-time orthogonal transform of a signal $z_k$ is defined as

$$Z_s^k = \sum_{r=-\infty}^{\infty} z_{k+r} w_r \psi(r, s) \, , \quad (11)$$

where $w_r$ is a window sequence, $\psi(r,s)$ represents the basis functions of an orthogonal transform. We use one-dimensional notation for simplicity. Equation (1) can be interpreted as the orthogonal transform of $z_{k+r}$ as viewed through the window $w_r$. $Z_s^k$ displays the orthogonal transform characteristics of the signal around time *k*. Note that while increased window length and resolution are typically beneficial in the spectral analysis of stationary data, for time-varying data it is preferable to keep the window length sufficiently. Assume that the window has finite length around $n$=0, and it is unity for all $r \in [-N_1, N_2]$. Here $N_1$ and $N_2$ are integer values. This leads to signal processing in a sliding window. In other words, local filters in the domain of an orthogonal transform at each position of a moving window modify the orthogonal transform coefficients of a signal to obtain only an estimate of the pixel $z_k$ of the window. The choice of orthogonal transform for sliding signal processing depends on many factors. The DCT is one the most appropriate transform with respect to the accuracy of power spectrum estimation from the observed data that is required for local filtering, the filter design, and computational complexity of the filter implementation. Linear filtering in the domain of DCT followed by inverse transforming is superior to that of the discrete Fourier transform (DFT) because a DCT can be considered as the DFT of a signal evenly extended outside its edges. This consequently attenuates boundary effects caused by circular convolution that are typical for linear filtering in the domain of DFT. First we define a local criterion of the performance of filters for image and signal processing and then derive optimal local adaptive filters with respect to the criterion. One the most used criterion in signal processing is the minimum mean-square error (MMSE). Since the processing is carried out in a moving window, then for each position of a moving window an estimate of the central element of the window is computed. Suppose that the signal to be processed is approximately stationary within the window. The signal may be distorted by sensor's noise. Let us consider a generalized linear filtering of a fragment of input one-dimensional signal (for instance for a fixed position of the moving window). Let $\boldsymbol{z}=[z_k]$ be undistorted real signal, $\boldsymbol{v}=[v_k]$ be an observed signal, $k=1,\ldots, N$, $N$ be the size of the fragment, $\boldsymbol{U}$ be the matrix of the discrete cosine transform, $E\{.\}$ be the expected value, superscript $^T$ denotes the transpose. Let $\bar{z} = \boldsymbol{H} v$ be a linear estimate of the undistorted signal, which minimizes the MMSE averaged over the window

$$MMSE = E\left\{\left(\boldsymbol{z}-\bar{\boldsymbol{z}}\right)^T\left(\boldsymbol{z}-\bar{\boldsymbol{z}}\right)\right\}\Big/N . \qquad (12)$$

The optimal filter for this problem is the Wiener filter [1]:

$$\boldsymbol{H} = E\left\{\boldsymbol{z}\,\boldsymbol{v}^T\right\}\left[E\left\{\boldsymbol{v}\boldsymbol{v}^T\right\}\right]^{-1}. \qquad (13)$$

Let us consider the known model of signal:

$$v_k = \sum_l a_{k,l} z_l + n_l , \qquad (14)$$

where $\boldsymbol{A}=[a_{k,l}]$ is a distortion matrix, $\boldsymbol{n}=[n_k]$ is additive noise with zero mean, $k,l=1,\ldots N$, $N$ is the size of fragment. The optimal filter is given by

$$\boldsymbol{H} = \boldsymbol{K}_{zz}\,\boldsymbol{A}^T\left[\boldsymbol{A}\boldsymbol{K}_{zz}\boldsymbol{A}^T + \boldsymbol{K}_{nn}\right]^{-1}, \qquad (15)$$

where $\boldsymbol{K}_{zz} = E\left\{\boldsymbol{z}\boldsymbol{z}^T\right\}$, $\boldsymbol{K}_{nn} = E\left\{\boldsymbol{n}\boldsymbol{n}^T\right\}$, $E\left\{\boldsymbol{z}\boldsymbol{n}^T\right\} = 0$ are the covariance matrices. It is assumed that an input signal and noise are uncorrelated. The obtained optimal filter is based on an assumption that an input signal within the window is stationary. The result of filtering is the restored window signal. This corresponds to signal processing in nonoverlapping fragments. Now suppose that the signal is processed in a moving window in the domain of the sliding DCT. For each position of the window an estimate of the central pixel should be computed. Using the equation for inverse sliding DCT presented in the previous section, the pointwise MSE for reconstruction of the central element of the window can be written as follows:

$$PMSE(k) = E\left\{\left[z(k)-\bar{z}(k)\right]^2\right\} = E\left\{\left[\sum_{l=1}^{N}\alpha(l)\left(Z(l)-\bar{Z}(l)\right)\right]^2\right\}, (16)$$

where $\bar{\boldsymbol{Z}} = \left[\bar{Z}(l) = H(l)V(l)\right]$ is a vector of signal estimate in the domain of the DCT, $\boldsymbol{H}_U = \left[H(l)\right]$ is a diagonal matrix of the scalar filter, $\boldsymbol{\alpha} = \left[\alpha(l)\right]$ is a diagonal matrix of the coefficients of inverse sliding cosine transform [10]. Minimizing (16), we obtain

$$\boldsymbol{H}_U = \left[\boldsymbol{P}_{vv}\right]^{-1}\boldsymbol{P}_{zv}\,\boldsymbol{I}_\alpha . \qquad (17)$$

where $\boldsymbol{P}_{zv} = \left[E\left\{Z(l)V(k)\right\}\right]$, $\boldsymbol{P}_{zz} = \left[E\left\{V(l)V(k)\right\}\right]$, $\boldsymbol{I_\alpha}$ is the identity matrix of the dimension of $\boldsymbol{\alpha}$. Note that matrix of coefficients $\boldsymbol{\alpha} = \left[\alpha(l)\right]$ for the inverse sliding transform is singular. The inverse sliding cosine transform possesses the dimension of the matrix twice less than the size of the window signal. Therefore, the computational complexity of the scalar filters in (17) and signal processing can be significantly reduced comparing to the complexity for the filter in (15). For the model of signal distortion in (14) the filter matrix is given as

$$\boldsymbol{H}_U = \left[\boldsymbol{U}\left(\boldsymbol{A}\boldsymbol{K}_{zz}\boldsymbol{A}^T + \boldsymbol{K}_{nn}\right)\boldsymbol{U}^T\right]^{-1}\boldsymbol{U}\,\boldsymbol{K}_{zz}\,\boldsymbol{A}^T\,\boldsymbol{U}^T\,\boldsymbol{I}_\alpha . (18)$$

If a signal has a high correlation coefficient and a smoothed version of the signal is corrupted by additive, weakly-correlated noise, then the matrix $\boldsymbol{U}\left(\boldsymbol{W}\boldsymbol{K}_{zz}\,\boldsymbol{W}^T + \boldsymbol{K}_{nn}\right)\boldsymbol{U}^T$ is close to diagonal. The linear convolution between a signal $\boldsymbol{x}$ and the matrix $\boldsymbol{K}_{zz}\,\boldsymbol{A}^T$ in the domain of the sliding DCT can be well approximated by a diagonal matrix $Diag\left(\boldsymbol{U}\boldsymbol{K}_{zz}\boldsymbol{A}^T\boldsymbol{U}^T\boldsymbol{I}_\alpha\right)\boldsymbol{V}$. Therefore, the matrix of the scalar filter in (18) is close to diagonal, and the filter can be written as

$$H(l) \approx \frac{P_1(l)}{P_2(l) + P_{nn}(l)} , \qquad (19)$$

where $P_1(l)$, $P_2(l)$, $P_{nn}(l)$ are diagonal elements of the following matrices $\boldsymbol{U}\,\boldsymbol{K}_{zz}\,\boldsymbol{A}^T\,\boldsymbol{U}^T\,\boldsymbol{I}_\alpha$, $\boldsymbol{U}\,\boldsymbol{A}\boldsymbol{K}_{zz}\boldsymbol{A}^T\,\boldsymbol{U}^T$, $\boldsymbol{U}\,\boldsymbol{K}_{nn}\,\boldsymbol{U}^T$, $l=1,\ldots N_1$, $N_1$ is the dimension of the matrix $\boldsymbol{I_\alpha}$. For the design of local adaptive filters in the domain of a sliding DCT the covariance matrices and power spectra of fragments of a signal are required. Since they are often unknown, in practice, these matrices can be estimated from observed signals [1], [9]. Next, computer simulation results for local adaptive restoration of images degraded by nonuniform motion blur is presented [16]. Assume that the blur is owing to horizontal relative motion between the camera and the image, and it is approximately space-invariant within local regions of the image. It is known that point spread functions for motion and focus blurs do have zeros in the frequency domain, and they can be uniquely identified by the location of these zero crossings [2]. We assume also that the observation noise is a zero-mean, white Gaussian process that is uncorrelated to the image signal. In this case, the noise field is completely characterized by its variance, which is commonly estimated by the sample variance computed over a low-contrast local region of the observed image. A real test aerial image is shown in Fig. 1(a). The size of image is 256x256, each pixel has 256 levels of quantization. The signal range is [0, 1]. The image quadrants are degraded by sliding 1D horizontal averaging with the following sizes of the moving window: 4, 3, 4, and 2 pixels (for quadrants from left to right, from top to bottom). The image is also corrupted by zero-mean additive white Gaussian noise. The degraded image with the noise standard deviation of 0.05 is shown in Fig. 1(b). Since there exists a difference in spectral distributions of the image signal and wide-band noise, the power spectrum of noise can be easily measured from the experimental covariance matrix. In our tests the window length of 15x15 pixels is used. The results of image restoration by the global parametric Wiener filtering [1] and the proposed method are shown in Figs. 1(c) and 1(d), respectively. Figs. 1(e) and 1(f) show a difference of the original image with the image restored by global Wiener algorithm, and the image restored with proposed algorithm, respectively.

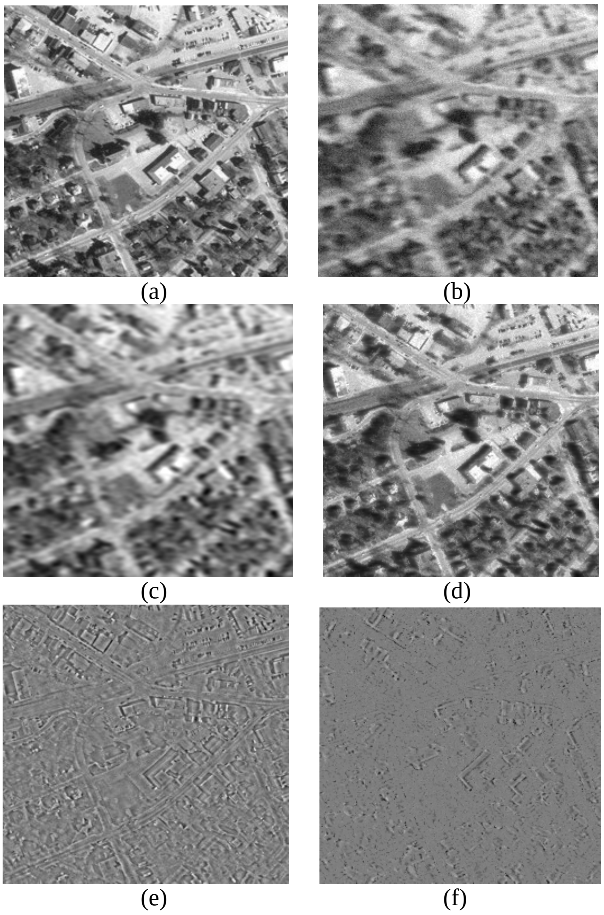


**Fig. 1. (a) Test image, (b) space-variant degraded test image, (c) global Wiener restoration, (d) local adaptive restoration in domain of sliding DCT, (e) difference between the original image and restored by global Wiener algorithm, (f) difference between the original image and restored by proposed algorithm.**

We see that the proposed algorithm is capable to perform a good space-variant image restoration and noise suppression. Finally, we investigate the robustness of the tested restoration techniques to additive noise. The performance of the global parametric Wiener filtering and the local adaptive filtering is shown in Fig. 2.

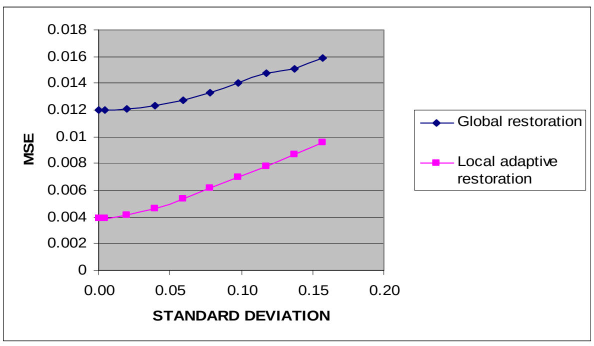


**Fig. 2. Performance of the restoration algorithms in terms of MSE versus the standard deviation of additive noise.**

## 5. CONCLUSION

In this paper, we presented a general concept of image restoration based on an appropriate regularization of variational integrals as well as an approximated solution of the restoration problem using local adaptive image processing. The PMSE estimator in the domain of sliding DCT is derived. To provide image processing at high rate, a fast recursive algorithm for computing the sliding DCT was utilized. Extensive testing using various parameters of degradations has shown that the original image can be well restored by proper choice of the algorithm parameters.

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