ADAPTATION ALGORITHMS AND DISTANCE ON EXPERTS' PROPOSITIONS IN MULTI-VALUED LOGICS

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Abstract: we consider statements of experts as logical formulas in an n-valued logic. Basing on model theory, we offer definitions of a metric on these statements along with a measure of their refutability and study their properties.

Keywords: statements of experts, distance measures, metrics

Introduction. Nowadays, there exists a great interest to building deciding functions basing on the analysis of expert information given in the form of logical probability statements of several experts using adaptation processes and coordination of statements [1-7]. We propose to consider expert propositions as formulas of the Lukasiewicz multi-valued logic. The suggested approach extends the case n = 3. The truth values of the formulas may be regarded as their possible probabilities. For organizing the search of logical patterns, both the distances between experts' propositions and the ones between the formulas in models (inside the database) at an arbitrary moment of time and the refutability measure are needed. A refutability measure enables us to range the expert statements basing on the level of their non-triviality or their importance. We should take into account that the state of a database and expert statements use to change in time. In the view of this, sometimes we will need to correct our database and experts' opinions to avoid explicit contradictions. The tools for such corrections (or refinement) of such a knowledge were developed by Vikentyev A.A., Lbov G.S., Koreneva L.N., Vikentiev R.A., and Novikov D.V. One can find a survey of such corrections for the class of logical decision functions in the papers [4-6] by G. S. Lbov and Gerasimov M. Here we do not consider these problems. The distances we introduce are based upon certain class of models; there exists a big variety of such classes. Experts could change this class of models and the theory itself; it is also possible to change them in case of obtaining wrong results. The cases n = 2 and n = 3 were studied earlier. Here we consider the case of an arbitrary n. Of course, not all of the results obtained earlier could be extended to the general case.

It is clear that different expert propositions and the related formulas could carry different amounts of information, and therefore a problem appears, how to range the statements according to this information and how to compare their self-descriptiveness. To solve these problems, we introduce the distance between any two arbitrary n-valued formulas. We also define the measure of refutability on these statements and study its properties.

The results of this paper could be generalized to an n-valued predicate logic with appropriate analogs for subsets of predicates corresponding to fixed truth values. This paper was supported by a RFFI 07-01-00331-a.

1. Definition of the distance between statements of experts

We assume the reader to be familiar with [1, 2].

Definition 1.1. The set $S^n(\varphi)$ of all elementary expressions contained in a formula φ of many-valued logic is called the support of φ.

Definition 1.2. For a set of formulas Σ, its support $S^n(\Sigma)$ is defined as $S^n(\Sigma) = \bigcup_{\varphi\in\Sigma} S^n(\varphi)$.

Definition 1.3. The set of all possible truth values of Σ is the set $Q^n(\Sigma) = \{\varphi_{\frac{k}{n-1}} \mid \varphi \in S(\Sigma),\ k = 1, \dots, n-1\}$.

Definition 1.4. A model M is a subset of $Q^n(\Sigma)$ which cannot contain both $\varphi_{\frac{k}{n-1}}$ and $\varphi_{\frac{l}{n-1}}$ simultaneously, $\forall k \neq l \quad \forall \varphi \in Q(\Sigma)$.

Now we formulate some properties of these concepts.

Let us define:

$$Mod_{S(\Sigma)}(A_{\frac{k}{n-1}}) = \{M \mid M \in P(S(\Sigma)),\ M \models A_{\frac{k}{n-1}}\}$$, and

$$Mod_{S(\Sigma)}(A_0) = \{M \mid M \in P(S(\Sigma)), M \not\models A_{\frac{k}{n-1}}, \forall k = 1, \dots, n-1\}$$

From the definition we derive:

1) $M \models (A \& B)_{\frac{k}{n-1}} \Leftrightarrow (M \models A_{\frac{p}{n-1}}$ and $M \models B_{\frac{q}{n-1}})$ $\min(p,q) = k$

2) $M \models (A \vee B)_{\frac{k}{n-1}} \Leftrightarrow (M \models A_{\frac{p}{n-1}}$ and $M \models B_{\frac{q}{n-1}})$ $\max(p,q) = k$

3) $M \models (\neg A)_{\frac{k}{n-1}} \Leftrightarrow M \models A_{\frac{n-1-k}{n-1}}$

In all the rest cases, the truth values will be equal to 0. Thus, for any truth value and any formula, there exists a class of all models on which this formula has the mentioned truth value.

Definition 1.5. Any two formulas are called equivalent if for all truth values they have the same classes of models, i.e. $\bigcup_{k=1}^{n-1} Mod_{S(\Sigma)}(\varphi)_{\frac{k}{n-1}} = \bigcup_{k=1}^{n-1} Mod_{S(\Sigma)}(\psi)_{\frac{k}{n-1}}$.

Here are some model-theoretic properties of these concepts.

Lemma 1.1. 1) $Mod_{S(\Sigma)}((A \& B)_{\frac{k}{n-1}}) =$

$$= \bigcup_{p=k}^{n-1} (Mod_{S(\Sigma)}(A)_{\frac{p}{n-1}} \cap Mod_{S(\Sigma)}(B)_{\frac{k}{n-1}}) \cup (Mod_{S(\Sigma)}(A)_{\frac{k}{n-1}} \cap Mod_{S(\Sigma)}(B)_{\frac{p}{n-1}});$$

2) $Mod_{S(\Sigma)}(A \vee B)\, d_{\frac{k}{n-1}} =$

$$= \bigcup_{p=0}^{k} \left(Mod_{S(\Sigma)}(A)\, d_{\frac{p}{n-1}} \cup Mod_{S(\Sigma)}(B)\, d_{\frac{k}{n-1}} \right) \cup \left(Mod_{S(\Sigma)}(A)\, d_{\frac{k}{n-1}} \cup Mod_{S(\Sigma)}(B)\, d_{\frac{p}{n-1}} \right);$$

3) $Mod_{S(\Sigma)}(\neg A)\, d_{\frac{k}{n-1}} = Mod_{S(\Sigma)}(A)\, d_{\frac{n-1-k}{n-1}}$;

4) $\bigcup_{k=1}^{n-1} Mod_{S(\Sigma)}(A)\, d_{\frac{k}{n-1}} = P(S(\Sigma)) \,/\, Mod_{S(\Sigma)}(\neg A)\, d_1$

Proposition 1.2. (on the number all of models $P^n(S(\Sigma))$ in $S^n(\Sigma)$) The number of models equals $|P(S(\Sigma))| = n^{|S(\Sigma)|}$.

Proof. By induction.

Definition 1.6 The distance between formulas φ and ψ such that $S(\varphi) \cup S(\psi) \subseteq S(\Sigma)$ on the set $P(S(\Sigma))$ is defined as (similar to case n=2)

$$\rho_{S(\Sigma)}(\varphi,\psi) = \frac{\left|\bigcup_{k=1}^{n-1} Mod_{S(\Sigma)}(\varphi\, d_{\frac{k}{n-1}} \,\&\, \psi\, d_0)\right| + \left|\bigcup_{k=1}^{n-1} Mod_{S(\Sigma)}(\varphi\, d_0 \,\&\, \psi\, d_{\frac{k}{n-1}})\right|}{n^{|S(\Sigma)|}}.$$

Theorem 1. For any formulas φ, the following is true

1) $0 \le \rho_{S(\Sigma)}(\varphi,\psi) \le 1$;

2) $\rho_{S(\Sigma)}(\varphi,\psi) = \rho_{S(\Sigma)}(\psi,\varphi)$;

3) $\rho_{S(\Sigma)}(\varphi,\psi) = 0 \Leftrightarrow \varphi \equiv \psi$;

4) $\rho_{S(\Sigma)}(\varphi,\psi) = 1 \Leftrightarrow \bigcup_{l=1}^{n-1}\bigcup_{k=1}^{n-1}\left(Mod(\varphi)_{\frac{k}{n-1}} \oplus Mod(\psi)_{\frac{l}{n-1}}\right) = P(S(\Sigma))$, where $\oplus$ is a direct union;

5) $\rho_{S(\Sigma)}(\varphi,\psi) \le \rho_{S(\Sigma)}(\varphi,\chi) + \rho_{S(\Sigma)}(\chi,\psi)$;

6) If $\varphi^1 \equiv \varphi^2$, then $\rho_{S(\Sigma)}(\varphi^1,\psi) = \rho_{S(\Sigma)}(\varphi^2,\psi)$.

Theorem 2. (Extension and preservation) For any φ, ψ from $S(\Sigma_0)$ and any $S(\Sigma_1)$ such that $S(\Sigma_0) \subset S(\Sigma_1)$ holds

$$\rho_{S(\Sigma_0)}(\varphi,\psi) = \rho_{S(\Sigma_1)}(\varphi,\psi).$$

2. Refutability

Definition 2.1 The measure of refutability of the formula φ from $\Phi(\Sigma) = \{\varphi \mid S(\varphi) \subset S(\Sigma)$ is the function

$$I_{S(\Sigma)}(\varphi) = \sum_{i=0}^{n-2} \alpha_i \frac{|M_{S(\Sigma)}(\varphi, d_{\frac{i}{n-1}})|}{n^{|S(\Sigma)|}},$$

where time-varying parameters α_i (part of the adaptation process) satisfy the conditions:

$$\begin{cases} 0 \le \alpha_i \le 1; \\ \alpha_i + \alpha_{n-1-i} = 1 \ \forall i = 0, \ldots, \frac{n-1}{2}; \\ \alpha_k \ge \alpha_i \ \forall k \le i. \end{cases}$$

Theorem 3. (Properties of refutability).

For any φ, $\psi \in \Phi(\Sigma)$ it is true that

1) $0 \le I_{S(\Sigma)}(\varphi) \le 1;$

2) $I_{S(\Sigma)}(\varphi) + I_{S(\Sigma)}(\neg\varphi) = 1;$

3) $I_{S(\Sigma)}(\varphi \,\&\, \psi) \ge \max(I_{S(\Sigma)}(\varphi), I_{S(\Sigma)}(\psi))$;

4) $I_{S(\Sigma)}(\varphi \vee \psi) \le \min(I_{S(\Sigma)}(\varphi), I_{S(\Sigma)}(\psi))$;

5) $I_{S(\Sigma)}(\varphi \vee \psi) + I_{S(\Sigma)}(\varphi \,\&\, \psi) = I_{S(\Sigma)}(\varphi) + I_{S(\Sigma)}(\psi)$;

6) $I^3{}_{S(\Sigma)}(\varphi \,\&\, \psi) = \dfrac{I^3{}_{S(\Sigma)}(\varphi) + I^3{}_{S(\Sigma)}(\psi) + \rho^3{}_{S(\Sigma)}(\neg\varphi, \neg\psi)}{2}$;

7) $I^3{}_{S(\Sigma)}(\varphi \vee \psi) = \dfrac{I^3{}_{S(\Sigma)}(\varphi) + I^3{}_{S(\Sigma)}(\psi) - \rho^3{}_{S(\Sigma)}(\neg\varphi, \neg\psi)}{2}$.

In addition, we have proved some special properties of refutability for distances and measures in cases n = 3 and n = 2. It is worth to mention that the selection of appropriate n is actually a part of the adaptation process for calculations of the distance and refutability measures.

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