STATISTICAL MODELS AND DECISIONS IN AIRCRAFT SERVICE

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Abstract. Aircraft structures have many components. Maintaining high reliability for these structures generally requires that the individual structure components have extremely high reliability, even after long periods of time. One of the most important problems in the fatigue analysis and design of aircraft structures is the prediction of the fatigue crack growth in service. Available in-service inspection data for various types of aircraft indicate that the fatigue crack damage accumulation in service involves considerable statistical variability. The objectives of this paper are to (i) describe possible statistical models to deal with the crack growth variability, (ii) point out their applications.

1. Introduction

Prediction of fatigue crack growth in aircraft structure components has not been an easy task. This is mainly because the manner in which the various parameters, such as loads, material properties and crack geometries, affect the crack propagation is not clearly understood [2]. This, consequently, had led to a proliferation of hypotheses and laws for describing fatigue crack propagation [2, 6]. Most of these models are based on concepts of the continuum theory with the assumption that cracks propagate in an ideal continuum media. Actual metallic materials, however, are composed of random microstructure described by various microparameters, which can seriously affect the growth of a crack in these materials. As a result, the deterministic theories can only be accepted as an approximation of the actual random fatigue crack propagation process, which, broadly speaking, has 5 phases: 1) Dormant. There are no cracks in the materials; 2) Nucleation. The crack is initially formed; 3) Micro-crack growth; The crack grows rather haphazardly up to about 1 mm in length; 4) Macro-crack growth. The crack continues to propagate before its growth rate finally increases dramatically; 5) Failure. The component fails; this occurs very quickly, relative to the other phases, and can be ignored as a factor in determining reliability. Thus, statistical fatigue life of structural components of aircraft may be divided, in general, into three stages, namely, crack nucleation, small crack growth, and large crack growth.

We consider in this paper the problem of estimating the minimum time-to-nucleation (or warranty period) for a number of aircraft structure components, before which no cracks in materials occur, based on the results of previous warranty period tests on the structure components in question. If in a fleet of aircraft there are m of the individual structure components, operating independently, the length of time until the first crack initially formed in any of these components is of basic interest, and provides a measure of assurance concerning the operation of the components in question. This leads to the consideration of the following problem. Suppose we have observations $X_1, ..., X_n$ as the results of nucleation tests conducted on the components; suppose also that there are m components of the same kind to be put into future use, with times-to-nucleation $Y_1, ..., Y_m$. Then we want to be able to estimate, on the basis of $X_1, ..., X_n$, the shortest time-to-nucleation (warranty period) $Y_{(1)}$ among the components $Y_1, ..., Y_m$. In this paper, the problem of estimating $Y_{(1)}$, the smallest of a future sample of m observations from the underlying distribution, based on an observed sample from the same distribution, is considered. The results have direct application in reliability theory, where the time until the first failure in a group of m items in service provides a measure of assurance regarding the operation of the items.

If, after warranty period, a micro-crack in aircraft structure components is not detected, the usual approach is to inspect the structures periodically at certain intervals. In this paper, a simple approach is proposed for situations where it is difficult to quantify the costs associated with inspections and undetected micro-cracks. It allows one to find the inspection policies for detection of micro-cracks in critical structural components of aircraft under the assumptions that the parameter values of the underlying distributions are known with certainty as well as when such is not the case. Further, in just the same way, obtaining inspection schedules under macro-crack propagation is considered.

2. Estimation of warranty period for aircraft structure components

In this section, we consider the problem of estimating the minimum time-to-nucleation (or warranty period) for a number of aircraft structure components, before which no cracks in materials occur, based on the results of previous warranty period tests on the structure components in question.

Let $X_1 < X_2 < ... < X_r$ be the first r ordered observations of time-to-micro-crack (about 1 mm in length) for identical structural components of aircraft from a sample of size n from a two-parameter Weibull distribution with probability density function

$$f(x;\sigma,\gamma)=\frac{\gamma}{\sigma}\left(\frac{x}{\sigma}\right)^{\gamma-1}\exp[-(x/\sigma)^{\gamma}],\quad x\geq 0, \tag{1}$$

where the parameters σ and γ (σ>0, γ>0) are unknown. Two types of censoring are generally recognized. In Type I censoring, the time, when censoring occurs, is fixed, and the number of survivors at this time are random variables. In Type II censoring, which is of primary interest here, the number of survivors are fixed and X_r is a random variable. In Type II censoring, the likelihood may be written as follows:

$$L\propto\left(\frac{\gamma}{\sigma}\right)^{r}\left(\prod_{i=1}^{r}\left(\frac{x_i}{\sigma}\right)^{\gamma-1}\right)\exp\left(-\sum_{i=1}^{r}\left(\frac{x_i}{\sigma}\right)^{\gamma}\right)\left[\int_{x_r}^{\infty}\frac{\gamma}{\sigma}\left(\frac{x}{\sigma}\right)^{\gamma-1}\exp[-(x/\sigma)^{\gamma}]dx\right]^{n-r}$$

$$\propto\left(\frac{\gamma}{\sigma^{\gamma}}\right)^{r}\left(\prod_{i=1}^{r}x_i^{\gamma-1}\right)\exp\left(-\frac{1}{\sigma^{\gamma}}\left[\sum_{i=1}^{r}x_i^{\gamma}+(n-r)x_r^{\gamma}\right]\right), \tag{2}$$

$$\ln L=\text{constant}+r(\ln\gamma-\gamma\ln\sigma)+(\gamma-1)\sum_{i=1}^{r}\ln x_i-\frac{\sum_{i=1}^{r}x_i^{\gamma}+(n-r)x_r^{\gamma}}{\sigma^{\gamma}}. \tag{3}$$

This leads to the likelihood equations

$$\frac{\partial\ln L}{\partial\sigma}=-\frac{r\gamma}{\sigma}+\frac{\sum_{i=1}^{r}x_i^{\gamma}+(n-r)x_r^{\gamma}}{\sigma^{2\gamma}}\gamma\sigma^{\gamma-1}=0, \tag{4}$$

$$\frac{\partial\ln L}{\partial\gamma}=\frac{r}{\gamma}-r\ln\sigma+\sum_{i=1}^{r}\ln x_i-\frac{\left(\sum_{i=1}^{r}x_i^{\gamma}\ln x_i+(n-r)x_r^{\gamma}\ln x_r\right)\sigma^{\gamma}-\left(\sum_{i=1}^{r}x_i^{\gamma}+(n-r)x_r^{\gamma}\right)\sigma^{\gamma}\ln\sigma}{\sigma^{2\gamma}}=0. \tag{5}$$

Then the MLE's $\widehat{\sigma}$ and $\widehat{\gamma}$ are solutions of

$$\widehat{\sigma}=\left(r^{-1}\left(\sum_{i=1}^{r}x_i^{\widehat{\gamma}}+(n-r)x_r^{\widehat{\gamma}}\right)\right)^{1/\widehat{\gamma}},\quad \widehat{\gamma}=\left[\left(\sum_{i=1}^{r}x_i^{\gamma}\ln x_i+(n-r)x_r^{\gamma}\ln x_r\right)\left(\sum_{i=1}^{r}x_i^{\gamma}+(n-r)x_r^{\gamma}\right)^{-1}-\frac{1}{r}\sum_{i=1}^{r}\ln x_i\right]^{-1}. \tag{6}$$

The variable lnX follows the extreme-value distribution,

$$f(\ln x;b,c)=\frac{1}{c}\exp\left(\frac{\ln x-b}{c}\right)\exp\left(-\exp\left(\frac{\ln x-b}{c}\right)\right),\quad -\infty<\ln x<\infty, \tag{7}$$

where b = lnσ and $c=\gamma^{-1}$. Now (7) is a distribution with location and scale parameters b and c, and it is well known that if $\widehat{b},\widehat{c}$ are maximum likelihood estimates for b, c from a complete sample of size n, then $(\widehat{b}-b)/c$, $(\widehat{b}-b)/\widehat{c}$ and $\widehat{c}/c$ are quantities whose distributions depend only on n.

In the present paper, we are interested in estimating Y_1, the smallest order statistic in a future sample of size m from the distribution (1). Now, it is easily shown that

$$V=(\ln Y_1-\widehat{b})/\widehat{c}=\widehat{\gamma}(\ln Y_1-\ln\widehat{\sigma})=\widehat{\gamma}\ln\left(\frac{Y_1}{\widehat{\sigma}}\right) \tag{8}$$

is parameter-free, with distribution depending only on n and m. Hence, probability statements for V lead to confidence interval statements for Y_1. Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ represent ordered observations. In particular, let $X_1 < X_2 < \dots < X_r$ be the first r ordered observations from a sample of size n from the distribution (1), i.e., we deal with Type II censoring. It can be shown, using the invariant embedding technique [3-5], that

$$\Pr\{V>v\}=\Pr\left\{\widehat{\gamma}\ln\left(\frac{Y_1}{\widehat{\sigma}}\right)>v\right\}=\frac{\int_0^{\infty}s^{r-2}e^{s\widehat{\gamma}\sum_{i=1}^{r}\ln(x_i/\widehat{\sigma})}\left(me^{sv}+\sum_{i=1}^{r}e^{s\widehat{\gamma}\ln(x_i/\widehat{\sigma})}+(n-r)e^{s\widehat{\gamma}\ln(x_r/\widehat{\sigma})}\right)^{-r}ds}{\int_0^{\infty}s^{r-2}e^{s\widehat{\gamma}\sum_{i=1}^{r}\ln(x_i/\widehat{\sigma})}\left(\sum_{i=1}^{r}e^{s\widehat{\gamma}\ln(x_i/\widehat{\sigma})}+(n-r)e^{s\widehat{\gamma}\ln(x_r/\widehat{\sigma})}\right)^{-r}ds}. \tag{9}$$

Now the probability statement

$$\Pr\left\{\hat{\gamma}\ln\left(\frac{Y_1}{\hat{\sigma}}\right) > v\right\} = 1-\alpha \tag{10}$$

leads to the warranty period $(0, \hat{\sigma}\exp(v/\hat{\gamma}))$ with confidence level $1-\alpha$.

3. Obtaining inspection schedules to detect micro-crack

Suppose an inspection is carried out at time t, and this shows that crack has not yet occurred. We now have to schedule the next inspection. Let X be the random time at which the crack may be occurred. Then we schedule the next inspection at time u>t, where u satisfies

$$\Pr\{X > u; X > t\} = 1-\alpha. \tag{11}$$

Equation (11) says that the next inspection is scheduled so that, with probability 1-α, the aircraft structure component is still working and free of micro-crack prior to inspection.

The inspection times (u_1, u_2, …) can be calculated recursively as follows. Let F(u) be the cumulative distribution function of the time-to-micro-crack. Then (11) says that

$$\frac{\overline{F}(u_{j+1})}{\overline{F}(u_j)} = 1-\alpha, \quad j \ge 0, \tag{12}$$

where $\overline{F}(u) = 1 - F(u).$ It can be shown that (12) is equivalent to the equation

$$1 - \frac{\overline{F}(u_{j+1})}{\overline{F}(u_j)} = \frac{\overline{F}(u_j) - \overline{F}(u_{j+1})}{\overline{F}(u_j)} = \frac{1-F(u_j) - [1-F(u_{j+1})]}{1-F(u_j)} = \frac{F(u_{j+1}) - F(u_j)}{1-F(u_j)} = \alpha, \quad j \ge 0, \tag{13}$$

that is, in other words, the probability that the micro-crack occurs in the time interval (u_j,u_{j+1}) without micro-crack at time u_j is always assumed α. It follows from (12) that

$$\overline{F}(u_{j+1}) = (1-\alpha)\overline{F}(u_j), \quad j \ge 0. \tag{14}$$

With u_0=0, u_1, u_2, … can be calculated recursively from (14). So that:

$$\overline{F}(u_j) = (1-\alpha)^j, \quad j = 1, 2, 3, \ldots, \tag{15}$$

the time u_j (j=1, 2, 3, ...) is given by

$$u_j = \overline{F}^{-1}[(1-\alpha)^j], \quad j = 1, 2, 3, \ldots. \tag{16}$$

Let N be the random number of inspections until the micro-crack occurs. Then

$$\Pr\{N \le j\} = \Pr\{X \le u_j\} = F(u_j), \tag{17}$$

$$E\{N\} = \sum_{j=0}^{\infty} j\Pr\{N = j\} = \sum_{j=1}^{\infty} j[\Pr\{N > j-1\} - \Pr\{N > j\}] = \sum_{j=0}^{\infty} P\{N > j\} = \sum_{j=0}^{\infty} \overline{F}(u_j) = \alpha^{-1}. \tag{18}$$

For example, if α=0.05 then, from (18), on average 20 inspections will be necessary.

When the time-to-micro-crack obeys the Weibull distribution (1) with unknown parameters σ and γ, then the inspection times can be calculated recursively from

$$\Pr\{V > v_{j+1}; V > v_j\} = \frac{\Pr\{V > v_{j+1}\}}{\Pr\{V > v_j\}} = \frac{\Pr\left\{\hat{\gamma}\ln\left(\frac{u_{j+1}}{\hat{\sigma}}\right) > v_{j+1}\right\}}{\Pr\left\{\hat{\gamma}\ln\left(\frac{u_j}{\hat{\sigma}}\right) > v_j\right\}}$$

$$= \frac{\int_0^{\infty} s^{r-2} e^{s\hat{\gamma}\sum_{i=1}^{r}\ln(x_i/\hat{\sigma})}\left(me^{sv_{j+1}} + \sum_{i=1}^{r} e^{s\hat{\gamma}\ln(x_i/\hat{\sigma})} + (n-r)e^{s\hat{\gamma}\ln(x_r/\hat{\sigma})}\right)^{-r} ds}{\int_0^{\infty} s^{r-2} e^{s\hat{\gamma}\sum_{i=1}^{r}\ln(x_i/\hat{\sigma})}\left(me^{sv_j} + \sum_{i=1}^{r} e^{s\hat{\gamma}\ln(x_i/\hat{\sigma})} + (n-r)e^{s\hat{\gamma}\ln(x_r/\hat{\sigma})}\right)^{-r} ds} = 1-\alpha, \quad j \ge 1. \tag{19}$$

4. Statistical models for estimating a time-to-failure for structural components of aircraft

The purpose of this section is to develop statistical models to estimate a time-to-failure for structural components of aircraft. The following examples provide some illustrations of degradation path models that lead to closed-form expressions for the cdf of the time-to-failure distribution.

Example 1. Suppose that the actual degradation path of a particular component is given by $\varphi(t)$ = a + Wt, where a is fixed and W varies from component to component according to a Weibull distribution

$$F(w;\sigma,\gamma)=\begin{cases}1-\exp[-(w/\sigma)^{\gamma}], & w\geq 0,\\ 0, & \text{otherwise.}\end{cases} \quad (20)$$

The parameter a represents the common initial amount of degradation of all the test components at the beginning of the test, $\varphi(0)=a$, and W represents degradation rate. We assume that the component degrades monotonically in time and φ is an increasing function, so $\Pr\{W > 0\} = 1$.

For the critical level h for the degradation path above which failure is assumed to have occurred, we can write h=a+WT, and then the failure time T = (h–a)/W. The distribution function of T is

$$G(t)=\Pr\{T\leq t\}=\Pr\left\{\frac{h-a}{W}\leq t\right\}=\Pr\left\{W\geq\frac{h-a}{t}\right\}=1-F\left(\frac{h-a}{t};\sigma,\gamma\right)=\exp\left[-\left(\frac{h-a}{\sigma t}\right)^{\gamma}\right],\quad t>0. \quad (21)$$

So the distribution function G(t) depends on a, h, and distribution parameters σ, γ. The distribution of T is known as the reciprocal Weibull because 1/T follows a Weibull distribution.

Similarly, if W follows a lognormal distribution (μ,σ^2), then

$$G(t)=\Pr\{T\leq t\}=\Phi\left(\frac{\log t-[\log(h-a)-\mu]}{\sigma}\right),\quad t>0. \quad (22)$$

where $\Phi(\cdot)$ is the standard normal distribution function. This shows that T follows a lognormal distribution.

Example 2. Suppose that a component path is given by $\varphi(t) = a_1 + W\exp(a_2t)$, $a_2 > 0$, where $\mathbf{a} = (a_1,a_2)'$ are fixed and W ~ lognormal(μ, σ^2), so

$$F(w;\mu,\sigma^2)=\Pr\{W\leq w\}=\Phi\left(\frac{\log w-\mu}{\sigma}\right),\quad w>0. \quad (23)$$

Then T can be expressed as follows:

$$T=\frac{\log(h-a_1)-\log W}{a_2}. \quad (24)$$

The distribution function of T for the critical level h is

$$G(t)=\Pr\{T\leq t\}=\Phi\left(\frac{t-[\log(h-a_1)-\mu]/a_2}{\sigma/a_2}\right),\quad t>0. \quad (25)$$

Therefore, we have

$$T\sim N\left(\frac{\log(h-a_1)-\mu]}{a_2},\frac{\sigma^2}{a_2^2}\right). \quad (26)$$

The possibility of negative T arises because, if $X > h - a_1$, then $\varphi(0) > h$ and $\varphi(t)$ crosses h before time 0.

5. Obtaining inspection schedules under macro-crack propagation

Let a(t) be the crack size of a fastener hole at t flight-hours, and a(0) be the initial macro-crack size at t=0. Laboratory test data and analytical investigations [1,7-9] indicate that the crack growth rate for specimens under design loading spectra may be represented by

$$\frac{da(t)}{dt}=Q[a(t)]^{B}, \quad (27)$$

in which Q and B are parameters depending on loading spectra, structural/material properties, etc. Integrating eqn (27) from t=0 to $t=\tau$, one obtains the relation between the crack size, $a(\tau)$, at any service time τ and the initial crack size, a(0), as follows

$$a(\tau)=\frac{a(0)}{\left[1-[a(0)]^{B-1}(B-1)Q\tau\right]^{1/(B-1)}}. \quad (28)$$

For the special case in which B=1, it can easily be shown that

$$a(\tau)=a(0)\exp(Q\tau). \quad (29)$$

Available in-service inspection data for various types of aircraft [7-8] indicate that the lognormal distribution provides a reasonable fit for B and Q in both cases. In this paper, for the sake of simplicity but without loss of generality, only a special case in which B=1 is considered. This suggests, by taking logs, the following model

$$\log[a(\tau)] = \log[a(0)] + W\tau, \quad (30)$$

where W=Q. We use a model

$$\log[a(t)] = \log[a(\tau)] + W(t-\tau), \quad (31)$$

where $t > \tau$ at given τ, and W follows a lognormal distribution with the cumulative distribution function

$$F(w;\mu,\sigma^2) = \Pr\{W \le w\} = \Phi\left(\frac{\log(w)-\mu}{\sigma}\right), \quad w \ge 0. \quad (32)$$

Let h_0 be the operational limit crack size for the degradation path, which is permitted for the initial macro-crack to grow and reach h_0, then we can write $\log(h_0)=\log[a(\tau)]+W(T-\tau)$, where $T-\tau = [\log(h_0)-\log[a(\tau)]]/W$ represents a time permitted for the initial macro-crack to grow and reach the operational limit crack size h_0. The distribution function of $T-\tau$ is given by

$$G(\Delta) = \Pr\{T-\tau \le \Delta\} = \Phi\left(\frac{\log(\Delta)-\log[\log(h_0/a(\tau))]-\mu]}{\sigma}\right), \quad t > 0. \quad (33)$$

Suppose an inspection is carried out at time τ, and this shows that $a(\tau)<h_0$. We now have to schedule the next inspection. Let T be the random time at which the $(a(T)=h_0)$ may be occurred. Then we schedule the next inspection at time $t > \tau$, where t satisfies

$$\Pr\{T > t; T > \tau\} = 1-\alpha. \quad (34)$$

Equation (34) says that the next inspection is scheduled so that, with probability $1-\alpha$, the aircraft structure component is still working and has the macro-crack size less than the operational limit crack size prior to inspection.

If u_0 and u_1 are known, the inspection times $(u_2, u_3, \ldots)$ can be calculated recursively as follows:

$$\frac{1-\Phi\left(\frac{\log(u_{j+1}-u_j)-\log[\log(h_0/a(u_j))]-\mu]}{\sigma}\right)}{1-\Phi\left(\frac{\log(u_j-u_{j-1})-\log[\log(h_0/a(u_{j-1}))]-\mu]}{\sigma}\right)} = 1-\alpha, \quad j \ge 1. \quad (35)$$

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