DYNAMIC AIRLINE SEAT INVENTORY CONTROL

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Abstract. Dynamic booking policy for multiple fare classes that share the same seating pool on one leg of an airline flight, when seats are booked in a nested fashion and when lower fare classes book before higher ones, is determined. The dynamic policy of airline booking makes repetitive use of a static method over the booking period, based on the most recent demand and capacity information. It allows one to allocate seats dynamically and anticipatory over time.

1. Introduction

Airlines sell the same seat at different prices according to the time at which the reservation is made and other conditions, such as if a Saturday night is in the period of a round-trip, in an attempt to charge business customers more and vacationers less. Thus the same seat can be sold at different prices. The question then arises whether to offer seats at a relatively low price at a given time with a given number of seats remaining or to wait for the possible arrival of a higher paying customer. Seats in an airplane are divided into compartments (first class, second class, etc.). Seats in each compartment are offered at several fares. We do not consider the partition of the airplane into compartments. Assigning seats in the same compartment to different fare classes of passengers is a major problem of airline seat allocation. We seek an optimal policy that maximizes total expected revenue. In the typical analysis of this problem, the following simplifying assumptions are often made: (1) Lower-valued fare classes book before higher-valued fare classes. (2) The fare classes within a nest are ordered by fare value (highest ranked class has highest fare value). (3) There are no cancellations of bookings. (4) Demand among fare classes is independent (demand in one class does not contain information about demand in other classes). (5) A denied request is revenue lost to the airline, i.e., we do not consider that a passenger who is denied a request will buy a higher value ticket (passenger sell-up) or take another flight on the same airline.

The problem is usually considered in three stages according to increasing difficulty. First is the one-leg problem, which deals with one airplane for one takeoff and landing and ignores the potential revenue impact of other links of the passengers' itineraries. Second is the multihop problem, which deals with one airplane having multiple takeoffs and landings (still ignoring the impact of other links). The third is the origin-destination network (OD network) problem, which considers many airplanes having many takeoffs and landings on a routing network. A common special case of the latter is the popular hub-and-spoke configuration.
By Assumption (1), passengers of lower paying fare classes come before passengers of higher fare classes. This gives rise to a conflict. If reservations are accepted from too many of the early arriving customers, often vacationers and other discretionary passengers, there will be too little room left for the higher paying--typically business--passengers who often book at the last moment. On the other hand, if too few reservations are accepted, there may not be enough higher paying passengers arriving later, which could result in empty seats when the plane leaves. This paper deals with the one-leg problem under the above assumptions.

2. Static policy of airline booking

Littlewood [5] was the first to propose a solution method for the seat inventory control problem for a single leg flight with two fare classes. The idea of his scheme is to equate the marginal revenues in each of the two fare classes. He suggests closing down the low fare class when the certain revenue from selling another low fare seat is exceeded by the expected revenue of selling the same seat at the higher fare. That is, low fare booking requests should be accepted as long as

$$c_2 \geq c_1 \Pr\{X_1 > u_1\}, \tag{1}$$

where c_1 and c_2 are the high and low fare levels respectively, X_1 denotes the demand for the high fare class, u_1 is the number of seats to protect for the high fare class and $\Pr\{X_1>u_1\}$ is the probability of selling all protected seats to high fare passengers. The smallest value of u_1 that satisfies the above condition is the number of seats to protect for the high fare class, and is known as the protection level of the high fare class. The concept of determining a protection level for the high fare class can also be seen as setting a booking limit, a maximum number of bookings, for the lower fare class. Both concepts restrict the number of bookings for the low fare class in order to accept bookings for the high fare class.

Richter [10] gave a marginal analysis, which proved that (1) gives an optimal allocation (assuming certain continuity conditions). Belobava [1] proposed a generalization of (1) to more than two fare classes called the Expected Marginal Seat Revenue (EMSR) method. In his approach, which is known as the EMSRa method, the protection level for the highest fare class u(1) is obtained from

$$c_2 = c_1 \Pr\{X_1 > u(1)\}. \tag{2}$$

This is just Littlewood's rule expressed as an equation, and it is appropriate as long as it is reasonable to approximate the protection level with a continuous variable and to attribute a probability density to the demand X_1. The total protection level for the two highest fare classes u(2) is obtained from

$$u(2) = u_1 + u_2, \tag{3}$$

where u_1 and u_2 are two individual protection levels determined from

$$c_3 = c_1 \Pr\{X_1 > u_1\} \tag{4}$$

and

$$c_3 = c_2 \Pr\{X_2 > u_2\}. \tag{5}$$

The total (nested) protection level for the three highest fare classes is obtained by summing three individual protection levels, and so on. This process is continued until nested protection levels, u(j), are obtained for all classes except the lowest. The booking limit for any class j is the just (U–u(j-1)), where U is the total number of seats available. The EMSRa method does, however, not yield optimal booking limits when more than two fare classes are considered.

Optimal policies for more than two classes have been presented independently by Curry [3], Wollmer [12], Brumelle and McGill [2], and Nechval et al. [8-9]. Curry uses continuous demand distributions and Wollmer uses discrete demand distributions. The approach Brumelle and McGill propose, is based on subdifferential optimization and admits either discrete or continuous demand distributions. They show that an optimal set of nested protection levels, u(1), u(2), . . . , u(m-1), where the fare classes are indexed from high to low, must satisfy the conditions:

$$\delta_+ E\{R_j(u(j))\} \le c_{j+1} \le \delta_- E\{R_j(u(j))\}, \tag{6}$$

for each j=1, 2, …, m-1, where $E\{R_j(u(j))\}$ is the expected revenue from the j highest fare classes when u(j) seats are protected for those classes and δ_+ and δ_- are the right and left derivatives with respect to u(j) respectively. These conditions express that a change in u(j) away from the optimal level in either direction will produce a smaller increase in the expected revenue than an immediate increase of c_{j+1}. The same conditions apply for discrete and continuous demand distributions. Notice, that it is only necessary to set m-1 nested protection levels when there are m fare classes on the flight leg, because no seats will have to be protected for the lowest fare class. Brumelle and McGill [2] show that under certain continuity conditions the conditions for the optimal nested protection levels reduce to the following set of probability statements:

$$\begin{gathered} c_2=c_1\Pr\{X_1>u(1)\}, \\ c_3=c_1\Pr\{X_1>u(1) \cap X_1+X_2>u(2)\}, \\ \vdots \\ c_m=c_1\Pr\{ X_1>u(1) \cap X_1+X_2>u(2) \cap \ldots \cap X_1+X_2 \cdots +X_{m-1}>u(m\text{-}1)\}. \end{gathered} \tag{7}$$

These statements have a simple and intuitive interpretation, much like Littlewood's rule. Just like Littlewood's rule and the EMSRa method, this method is based on the idea of equating the marginal revenues in the various fare classes and therefore belongs to the class of EMSR methods. The method is called the EMSRb method. In Nechval et al. [8] use is made of a technique of Lagrange multipliers, which admits continuous demand distributions and allows one to obtain results in the form suitable for a practical use. Robinson [11] finds the optimality conditions when the assumption of a sequential arrival order with monotonically increasing fares is relaxed into a sequential arrival order with an arbitrary fare order. Furthermore, Curry [3] provides an approach to apply his method to origin–destination itineraries instead of single flight legs, when the capacities are not shared among different origin–destinations.

3. Dynamic policy of airline booking

It will be noted that the solution methods described above are all static. This class of solution methods is optimal under the sequential arrival assumption as long as no change in the probability distributions of the demand is foreseen. However, information on the actual demand process can reduce the uncertainty associated with the estimates of demand. Hence, repetitive use of a static method over the booking period, based on the most recent demand and capacity information, is the general way to proceed.

In this section, we consider a flight for a single departure date with T predefined reading dates at which the dynamic policy is to be updated, i.e., the booking period before departure is divided into T readings periods determined by the T reading dates. These reading dates are indexed in decreasing order, t=T, …, 1, 0, where t=1 denotes the first interval immediately preceding departure, and t=0 is at departure. The T-th reading period begins at the initial reading date at the beginning of the booking period, and the t-th reading period begins at t-th reading date furthest from the departure date. Thus, the indexing of the reading periods counts downwards as time moves closer to the departure date. Typically, the reading periods that are closer to departure cover much shorter periods of time than those further from departure. For example, the reading period immediately preceding departure may cover 1 day whereas the reading period 1-month from departure may cover 1 week.

Let us suppose that the total seat demand for fare class j at the t-th reading date (time t) prior to flight departure is X_{jt} ($j\in\{1, 2, \dots , m\}$), where X_{1t} corresponds to the highest fare class; $f_{jt}(x_{jt};\theta_{jt})$ is the probability density function of X_{jt}, where θ_{jt} is a parameter (in general, vector). We assume that these demands are stochastically independent. The vector of demands is $\mathbf{X}_t=(X_{1t}, \dots , X_{mt})$. Each booking of a fare class j seat generates average revenue of c_j, where $c_1>c_2> \dots >c_m$. Let u_{jt}, $j\in\{1, \dots, m-1\}$ be an individual protection level for fare class j at time t prior to flight departure. This many seats are protected for class j from all lower classes. The protection for the two highest fare classes is obtained by summing two individual protection levels, $(u_{1t}+u_{2t})$, and so on. There is no protection level for the lowest fare class, m; u_{mt} is the booking limit, or number of seats available, for class m at time t prior to flight departure; class m is open as long as the number of bookings in class m remains less than this limit. Thus, $(u_{jt}+ \dots +u_{mt})$ is the booking limit, or number of seats available, for class j, $j\in\{1, \dots, m\}$. Class j is open as long as the number of bookings in class j and lower classes remain less than this limit. The maximum number of seats that may be booked by fare classes in the next at time t prior to flight departure is the number of unsold seats U_t. Demands for the lowest fare class arrive first, and seats are booked for this class until a fixed time limit is reached, bookings have reached some limit, or the demand is exhausted. Sales to this fare class are then closed, and sales to the class with the next lowest fare are begun, and so on for all fare classes. It is assumed that any time limits on bookings for fare classes are prespecified. That is, the setting of such time limits is not part of the problem considered here. It is possible, depending on the airplane capacity, fares, and demand distributions that some fare classes will not be opened at all.

Problem statement. Since the fare requests in each class are independent, we may find the expected revenue for m classes, $R_{mt}(u_{1t}, u_{2t}, \dots , u_{mt})$, in terms of the revenue for class m, plus the expected revenue of the remaining m-1 classes, accrued from reading period t to departure, given that U_t specifies the remaining set capacity at the beginning of reading period t. Thus, the problem at time t prior to flight departure is to find an optimal vector of individual protection levels (for the m-1 highest fare classes) and booking limit (for the lowest fare class m),

$$(u_{1t}^*, u_{2t}^*, \dots, u_{mt}^*) = \arg \max_{(u_{1t}, u_{2t}, \dots, u_{mt})\in D_t} R_{mt}(u_{1t}, u_{2t}, \dots, u_{mt}), \tag{8}$$

where

$$R_{mt}(u_{1t}, u_{2t}, ..., u_{mt}) = \int_0^{u_{mt}} [c_m x_{mt} + R_{m-1,t}(u_{t1}, ..., u_{m-2,t}, u_{m-1,t} + u_{mt} - x_{mt})] f_{mt}(x_{mt};\theta_{mt}) dx_{mt}$$

$$+ \int_{u_{mt}}^{\infty} [c_m u_{mt} + R_{m-1,t}(u_{1t}, ..., u_{m-1,t})] f_{mt}(x_{mt};\theta_{mt}) dx_{mt}, \tag{9}$$

is the expected revenue, with $R_{0t}(\cdot)=0$,

$$D_t = \left\{ (u_{1t}, ..., u_{mt}) : \sum_{j=1}^{m} u_{jt} = U_t, u_{jt} \ge 0, \forall j = 1(1)m \right\}. \tag{10}$$

Sufficiency conditions for constrained optima. The constrained optimization problem is defined by (8). The sufficiency conditions for constrained optima are given by the following theorem.

Theorem 1. Let

$$B_k = -\frac{1}{(J^\circ)^2} \begin{vmatrix} 0 & g_1 & \cdots & g_{1+k} \\ g_1 & L_{11} & \cdots & L_{1,1+k} \\ \vdots & \vdots & \vdots & \vdots \\ g_{1+k} & L_{1+k,1} & \cdots & L_{1+k,1+k} \end{vmatrix}, \; k=1(1)m-1, \tag{11}$$

where

$$L = R_{mt}(u_{1t}, u_{2t}, ..., u_{mt}) + \lambda g \tag{12}$$

is the Lagrangian function, λ is the Lagrange multiplier,

$$g = U_t - \sum_{j=1}^{m} u_{jt}, \tag{13}$$

$$L_{ij} = \partial^2 L / \partial u_{it} \partial u_{jt}, \quad g_i = \partial g / \partial u_{it}, \quad i, j = 1, ..., m, \tag{14}$$

and

$$J^{\circ} = \frac{\partial g}{\partial u_{1t}} = -1 \neq 0. \tag{15}$$

Then, for a maximum of $R_{mt}(u_{1t}, u_{2t}, \ldots, u_{mt})$, the B_k should alternate in sign, B_1 being negative.

Proof. The proof follows immediately by using the result of [4] or [6-7]. □

An optimal set of individual protection levels $(u^*_{1t}, u^*_{2t}, ..., u^*_{m-1,t})$ must satisfy the conditions given by the following theorem.

Theorem 2. Under conditions given by Theorem 1, the optimal protection levels can be obtained by finding $u^*_{1t}, u^*_{2t}, ..., u^*_{m-1,t}$ that satisfy

$$c_2 = c_1 \int_{u^*_{1t}}^{\infty} f_{1t}(x_{1t};\theta_{1t})dx_{1t},$$

$$c_3 = c_2 \int_{u^*_{2t}}^{\infty} f_{2t}(x_{2t};\theta_{2t})dx_{2t} + c_1 \int_{0}^{u^*_{2t}} f_{2t}(x_{2t};\theta_{2t}) \int_{u^*_{1t}+u^*_{2t}-x_{2t}}^{\infty} f_{1t}(x_{1t};\theta_{1t})dx_{1t}dx_{2t},$$

$$c_4 = c_3 \int_{u^*_{3t}}^{\infty} f_{3t}(x_{3t};\theta_{3t})dx_{3t} + c_2 \int_{0}^{u^*_{3t}} f_{3t}(x_{3t};\theta_{3t}) \int_{u^*_{2t}+u^*_{3t}-x_{3t}}^{\infty} f_{2t}(x_{2t};\theta_{2t})dx_{2t}dx_{3t} + c_1 \int_{0}^{u^*_{3t}} f_{3t}(x_{3t};\theta_{3t}) \int_{0}^{u^*_{2t}+u^*_{3t}-x_{3t}} f_{2t}(x_{2t};\theta_{2t})$$

$$\times \int_{u^*_{1t}+u^*_{2t}+u^*_{3t}-x_{3t}-x_{2t}}^{\infty} f_{1t}(x_{1t};\theta_{1t})dx_{1t}dx_{2t}dx_{3t},$$

$$\vdots$$

$$c_k = c_{k-1} \int_{u^*_{k-1,t}}^{\infty} f_{k-1,t}(x_{k-1,t};\theta_{k-1,t})dx_{k-1,t} + c_{k-2} \int_{0}^{u^*_{k-1,t}} f_{k-1,t}(x_{k-1,t};\theta_{k-1,t}) \int_{u^*_{k-2,t}+u^*_{k-1,t}-x_{k-1,t}}^{\infty} f_{k-2,t}(x_{k-2,t};\theta_{k-2,t})dx_{k-2,t}dx_{k-1,t}$$

$$+ \ ... \ + c_1 \int_{0}^{u^*_{k-1,t}} f_{k-1,t}(x_{k-1,t};\theta_{k-1,t}) \int_{0}^{u^*_{k-2,t}+u^*_{k-1,t}-x_{k-1,t}} f_{k-2,t}(x_{k-2,t};\theta_{k-2,t}) \cdots \int_{0}^{u^*_{2t}+...+u^*_{k-1,t}-x_{k-1,t}-...-x_{3t}} f_{2t}(x_{2t};\theta_{2t})$$

$$\times \int_{u^*_{1t}+u^*_{2t}+...+u^*_{k-1,t}-x_{k-1,t}-...-x_{3t}-x_{2t}}^{\infty} f_{1t}(x_{1t};\theta_{1t}) \ dx_{1t}...dx_{k-1,t}, \tag{16}$$

where $k \in \{2, \ldots, m-1\}$.

Proof. The proof is a simple application of the Lagrange multipliers technique. □

One can see that the above equations are solved recursively for each fare class starting with the first fare class. This process is continued until we have the first k such that

$$\sum_{j=1}^{k-1} u^*_{jt} \leq U_t \text{ and } \sum_{j=1}^{k} u^*_{jt} > U_t, \quad u^*_{jt} > 0, \quad k \in \{2, ..., m-1\}. \tag{17}$$

Then

$$u^*_{kt} = \max\left(0, U_t - \sum_{j=1}^{k-1} u^*_{jt}\right), \quad k \in \{2, ..., m-1\}, \tag{18}$$

and $u^*_{j,t} = 0$ for all j>k. Otherwise, if

$$\sum_{j=1}^{m-1} u^*_{jt} \leq U_t, \tag{19}$$

then the optimal booking limit for the lowest fare class, m, is

$$u_{mt}^{*} = \max\left(0, U_t - \sum_{j=1}^{m-1} u_{jt}^{*}\right). \tag{20}$$

It follows from the above that, in general, an optimal set of individual protection levels must satisfy the following conditions:

$$\begin{aligned}
c_2 &= c_1 \Pr\{X_{1t} > u_{1t}^{*}\},\\
c_3 &= c_1 \Pr\{(X_{1t} > u_{1t}^{*}) \bigcap (X_{1t} + X_{2t} > u_{1t}^{*} + u_{2t}^{*})\}],\\
c_4 &= c_1 \Pr\{(X_{1t} > u_{1t}^{*}) \bigcap (X_{1t} + X_{2t} > u_{1t}^{*} + u_{2t}^{*})\\
&\quad \bigcap (X_{1t} + X_{2t} + X_{3t} > u_{1t}^{*} + u_{2t}^{*} + u_{3t}^{*})\},\\
&\vdots\\
c_k &= c_1 \Pr\{(X_{1t} > u_{1t}^{*}) \bigcap (X_{1t} + X_{2t} > u_{1t}^{*} + u_{2t}^{*})\\
&\quad \bigcap \ldots \bigcap (X_{1t} + X_{2t} + \ldots + X_{k-1,t} > u_{1t}^{*} + u_{2t}^{*} + \ldots + u_{k-1,t}^{*})\},
\end{aligned} \tag{21}$$

where $k \in \{2, \ldots, m\text{-}1\}$.

4. Conclusion

This paper considers the airline seat inventory control problem for a single leg route taking into account dynamics and uncertainty of booking process. We show that a booking policy that maximizes expected revenue can be characterized by a simple set of conditions that relate the probability distributions of demand for the various fare classes to their respective fares.

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