ON DEPARTURE FLOWS OF THE SERVED CUSTOMERS IN MULTIPHASE QUEUEING SYSTEMS

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Abstract. Multiphase queueing systems (tandem queues, queues in series) are of special interest both in theory and in practical applications (packet switch structures, cellular mobile networks, message switching systems, retransmission of video images, assembly lines, processes of conveyor production, etc.). In this paper, we deal with approximations of multiphase queueing systems. We investigated departure flows of the served customers under heavy traffic in multiphase queueing systems. The theorem on the law of the iterated logarithm for departure flows of the customers served under heavy traffic conditions in multiphase queueing systems has been proved.

1. Introduction

The paper is designated to the analysis of queueing systems, arising in the network theory and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). Thus, in this paper, we investigated departure flows of the served customers under heavy traffic in multiphase queueing systems. The theorem on the law of the iterated logarithm for departure flows of the customers served under heavy traffic conditions in multiphase queueing systems has been proved.

The works on departure time or number of departures for the queues in heavy traffic are sparse. One of the first papers of this kind [9], functional limit theorems for the number of departures and the departure time of the customer in single-server queues are proved. In [15] and [5], it is investigated departures in multiclass and multiserver queues. In [6], it is presented studies of simulations for a departure process in multiphase queueing systems. In [14], [1] and [5], it is investigated the limiting behaviour of the departure time of the customer in multiphase queueing systems. In [13], it is presented the convergence of departures on tandem queues. In [12], functional limit theorems for the number of departures in multiphase queueing systems for various conditions of heavy traffic are proved.

We are investigating here the k-phase queueing systems (see, for example, [10]). A queueing system consisting of k consecutive service units is said to be a k-phase queueing system. The service process at the ith service unit is called the ith phase of service. Upon completing its service at the ith service unit (i=1, … , k-1), a customer enters immediately the (i+1)st phase. Upon completing its service at the kth service unit, a customer leaves a system. Queues of unbounded length are allowed at any service unit: FCFS service discipline is assumed. Let us denote t_n as the time of arrival of the n-th customer; $S_n^{(j)}$ as the service time of the multiphase queueing systems; $z_n = t_{n+1} - t_n$. Let us introduce mutually independent renewal processes $x_j(t) = \left\{ \max_k \sum_{i=1}^{k} S_i^{(j)} \le t \right\}$ (such a total number of customers can be served in the jth phase of multiphase queueing systems until time t if devices are working without time wasted), $e(t) = \left\{ \max_k \sum_{i=1}^{k} z_i \le t \right\}$ (total number of customers which arrive at multiphase queueing systems until time t). Next, denote by $\tau_j(t)$ the total number of customers after service departure from the jth phase of multiphase queueing systems until time t; $Q_j(t)$ as the queue length of customers in the j-th phase of multiphase queueing systems at a time moment t; $v_j(t) = \sum_{i=1}^{j} Q_i(t)$ stands for the summary queue length of customers in the jth phase of multiphase queueing systems at the time moment t, j=1,2,… ,k and t>0. Suppose that the random variables are defined on one common probability space (Ω, F, P). Let interarrival times (z_n) to the multiphase queueing system and service times $(S_n^{(j)})$ in every phase of the multiphase queueing system for (j=1, 2,… ,k) be mutually independent identically distributed random variables. Let us define $\mu_j = (ES_1^{(j)})^{-1}$, $\mu_0 = (Ez_1)^{-1}$, $\alpha_j = \mu_0 - \mu_j$, $\alpha_0 \equiv 0$, $\tilde{\sigma}_j^2 = DS_1^{(j)} \cdot (ES_1^{(j)})^{-3} > 0$, $\tilde{\sigma}_0^2 = Dz_1 \cdot (Ez_1)^{-3} > 0$, $\sigma_j^2 = \tilde{\sigma}_j^2 + \tilde{\sigma}_{j-1}^2$, $\tilde{e}_j(t) = e(t) - x(t)$, $\tilde{x}_j(t) = x_{j-1}(t) - x_j(t)$, $j = 1,2, \ldots, k$ and $t > 0$.

In [11], the relations

$$Q_j(t) = \tau_{j-1}(t) - \tau_j(t), \tag{1}$$

$$Q_j(t) = f_t(\tau_{j-1}(\cdot) - x_j(\cdot)), \tag{2}$$

are obtained for (j=1, 2,…,k) and $f_t(x(\cdot)) = x(t) - \inf_{0 \le s \le t} x(s)$.

Next, using (1) - (2), we achieve that

$$\tau_j(t) = \tau_{j-1}(t) - Q_j(t) = x_j(t) + \inf_{0 \le s \le t} (\tau_{j-1}(s) - x(s)), \text{ for } j=1, 2,\ldots,k \text{ and } \tau_0(t) = e(t). \tag{3}$$

Also, we note that

$$v_j(t) = e(t) - \tau_j(t), \quad j=1,2,\ldots,k. \tag{4}$$

At first assume the following condition to be fulfilled $\mu_0 > \mu_1 > \ldots > \mu_k > 0$.

Then

$$\alpha_k > \alpha_{k-1} > \ldots > \alpha_1 > 0. \tag{5}$$

2. Main results

One of the main results of the paper is a theorem on the law of the iterated logarithm for the departure flows of served customers in multiphase queueing systems.

Theorem. *If conditions* (5) *are fulfilled, then*

$$P\left(\varliminf_{n\to\infty} \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)} = -1 \right) = P\left(\varlimsup_{n\to\infty} \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)} = 1 \right) = 1,$$

j=1,2,…,k, $0 \le t \le 1$ and $a(n) = \sqrt{2n \ln \ln n}$.

Proof. First, denote

$$x_j^n(t) = \frac{x_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)}, \tau_j^n(t) = \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)}, \ j = 1,2,\ldots,k, 0 \le t \le 1.$$

We note that (see (3))

$$x_j(t) - \tau_j(t) = \sup_{0 \le s \le t} (x_j(s) - \tau_{j-1}(s)), \quad j = 1,2,\ldots,k \text{ and } t > 0.$$

Thus,

$$x_j(nt) - \tau_j(nt) = \sup_{0 \le s \le nt} (x_j(s) - \tau_{j-1}(s)) = \sup_{0 \le s \le t} (x_j(ns) - \tau_{j-1}(ns)), \quad j = 1,2,\ldots,k \text{ and } t > 0.$$

From this we get for arbitrary $\varepsilon > 0$ that

$$P(\rho(x_j^n, \tau_j^n) > \varepsilon) = P\left(\sup_{0 \le t \le 1} (x_j(nt) - \tau_j(nt)) > \varepsilon \cdot \sigma_j \cdot a(n) \right) \le$$

$$P\left(\sup_{0 \le t \le 1} (\sup_{0 \le s \le t} (x_j(ns) - \tau_{j-1}(ns))) > \varepsilon \cdot \sigma_j \cdot a(n) \right) \le \ P\left(\sup_{0 \le t \le 1} (x_j(nt) - \tau_{j-1}(nt)) > \varepsilon \cdot \sigma_j \cdot a(n) \right) \le$$

$$P\left(\sup_{0 \le t \le 1} (x_j(nt) - x_{j-1}(nt)) > \frac{\varepsilon}{2} \cdot \sigma_j \cdot a(n) \right) + \tag{6}$$

$$P\left(\sup_{0\le t\le 1}(x_{j-1}(nt)-\tau_{j-1}(nt))>\frac{\varepsilon}{2}\cdot\sigma_j\cdot a(n)\right)\le\ldots\le\sum_{i=1}^{j}P\left(\sup_{0\le t\le 1}(-\tilde{x}_i(nt))>\frac{\varepsilon}{j}\cdot\sigma_j\cdot a(n)\right)\le$$

$$\sum_{i=1}^{k}P\left(\sup_{0\le t\le 1}(-\tilde{x}_i(nt))>\frac{\varepsilon}{j}\cdot\sigma_j\cdot a(n)\right)\le\sum_{i=1}^{k}P\left(\sup_{0\le t\le 1}(-\tilde{x}_j(nt))>\frac{\varepsilon}{k}\cdot\sigma_j\cdot a(n)\right),\quad j=1,2,\ldots,k.$$

We achieve that, if conditions (5) are fulfilled, then (see, for example, [8])

$$\frac{\sup_{0\le t\le 1}(-\tilde{x}_j(nt))}{a(n)}\Rightarrow 0\,,\ \ j=1,2,\ldots,k\,. \tag{7}$$

But, applying the law of the iterated logarithm to the renewal process (see [7]), we obtain

$$P\left(\underline{\lim}_{n\to\infty}\,x_j^n(t)=-1\right)=P\left(\overline{\lim}_{n\to\infty}\,x_j^n(t)=1\right)=1\,,\ \ j=1,2,\ldots,k\ \text{ and }\ 0\le t\le 1. \tag{8}$$

From this, (6) and (7) we get that

$$P\left(\underline{\lim}_{n\to\infty}\,\tau_j^n(t)=-1\right)=P\left(\overline{\lim}_{n\to\infty}\,\tau_j^n(t)=1\right)=1\,,\ \ j=1,2,\ldots,k\ \text{ and }\ 0\le t\le 1.$$

The proof of the theorem is complete.

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