ANALYTICAL METHOD FOR RESEARCH OF THE QUASI-STATIONARY HEAT TRANSFER PROCESS WITH THE MIXED BOUNDARY CONDITIONS

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Abstract. For an axially symmetric two-dimensional non-stationary heat transfer problem with the mixed boundary conditions the new method for research of coming quasi-stationary phases is offered. By obviously way the results of such research may be put in a basis of development of identification methods of heat characteristics and tools of non-destroying monitoring of heat transfer process. The analytical solution for research of a quasi-stationary heat transfer phase on model of the isotropic half-space which are heated up through circular area on its surface is obtained. Heating is carried out as follows: inside a circle on a surface of a half-space arbitrary function of exterior temperature is given, and outside of this circle the ideal heat isolation of a surface is exists.

In a cylindrical coordinates (r,z,τ) mathematical form of corresponding axis-symmetrical problem with a temperature function $T(r,z,\tau)$ can be noted as follows:

$$T_{rr}(r,z,\tau)+\frac{1}{r}T_r(r,z,\tau)+T_{zz}(r,z,\tau)=\frac{1}{a}T_\tau(r,z,\tau), \qquad r>0, z>0, \tau>0, \tag{1}$$

$$T(r,z,0)=T_0, \quad r\ge 0, \quad z\ge 0, \tag{2}$$

$$\frac{\partial T(0,z,\tau)}{\partial r}=0, \quad z\ge 0, \quad \tau\ge 0, \tag{3}$$

$$\frac{\partial T(\infty,z,\tau)}{\partial r}=\frac{\partial T(r,\infty,\tau)}{\partial z}=0, \; r\ge 0, \quad z\ge 0, \quad \tau\ge 0, \tag{4}$$

with the mixed boundary conditions

$$T(r,0,\tau)=f(r,\tau), \quad 0<r<R, \quad \tau>0, \tag{5}$$

$$T_z(r,0,\tau)=0, \quad R<r<\infty, \quad \tau>0. \tag{6}$$

By the using of a integral Laplace transform and by the introduction of a surplus temperature $\theta(r,z,\tau)=T(r,z,\tau)-T_0$ equations (1), (5) and (6) take a forms

$$\bar\theta_{rr}(r,z,s)+\frac{1}{r}\bar\theta_r(r,z,s)+\bar\theta_{zz}(r,z,\tau)=\frac{s}{a}\bar\theta(r,z,s), \quad r>0, \; z>0, \tag{7}$$

$$\bar\theta(r,0,s)=\bar f(r,s)-\frac{T_0}{s}, \quad 0<r<R, \tag{8}$$

$$\bar\theta_z(r,0,s)=0, \quad R<r<\infty, \quad \operatorname{Re}s>0, \tag{9}$$

where

$$\bar\theta(r,z,s)=\bar T(r,z,s)-\frac{T_0}{s}=\int_0^\infty \theta(r,z,\tau)\exp(-s\tau)d\tau, \tag{10}$$

$$\bar f(r,s)=\int_0^\infty f(r,\tau)\exp(-s\tau)d\tau. \tag{11}$$

The general solution of this problem has a form [1]

$$\bar\theta(r,z,s)=\int_0^\infty \bar C(p,s)\exp\left(-z\sqrt{p^2+\frac{s}{a}}\right)\mathrm{J}_0(pr)dp, \tag{12}$$

where $\mathrm{J}_0(pr)$ is a Bessel function first kind and zero order, $\bar C(p,s)$ is unknown analytical image-function.

According to mixed boundary conditions (8) and (9) to find $\bar C(p,s)$ we should solve the following dual integral equations with L-parameter ($\operatorname{Re}s>0$):

$$\int_0^\infty \bar C(p,s)\mathrm{J}_0(pr)dp=\bar f(r,s)-\frac{T_0}{s}, \quad 0<r<R, \tag{13}$$

$$\int_0^\infty \overline{C}(p,s)\sqrt{p^2+\frac{s}{a}}\ \mathrm{J}_0(pr)dp=0, \quad R<r<\infty. \tag{14}$$

By the using of substitution [2]

$$\overline{C}(p,s)=\frac{p}{\sqrt{p^2+\frac{s}{a}}}\int_0^R \overline{\varphi}(t,s)\cos\left(t\sqrt{p^2+\frac{s}{a}}\right)dt \tag{15}$$

the second dual integral equation (14) is fulfilled (with account of corresponding non-continuous integral on $R<r<\infty$) and from (13) it is need to solve the following integral equation on $0<r<R$:

$$\int_0^r \frac{\overline{\varphi}(t,s)}{\sqrt{r^2-t^2}}\exp\left(-\frac{\sqrt{r^2-t^2}}{\sqrt{a}}\sqrt{s}\right)dt-\int_r^R \frac{\overline{\varphi}(t,s)}{\sqrt{t^2-r^2}}\sin\left(\frac{\sqrt{t^2-r^2}}{\sqrt{a}}\sqrt{s}\right)dt=\overline{f}(r,s)-\frac{T_0}{s}. \tag{16}$$

Let's present function $\overline{\varphi}(t,s)$ as a series [3]

$$\overline{\varphi}(t,s)=\frac{1}{s}\exp\left(-\frac{2R}{\sqrt{a}}\sqrt{s}\right)\sum_{n=0}^{\infty}\varphi_n(t)\left(\sqrt{s}\right)^n. \tag{17}$$

Substitution $\overline{\varphi}(t,s)$ from (17) into (16) reduces in the equation:

$$\begin{aligned}&\int_0^r \frac{1}{\sqrt{r^2-t^2}}\frac{1}{\sqrt{s}}\exp\left(-\frac{2R}{\sqrt{a}}\sqrt{s}\right)\sum_{n=0}^{\infty}\varphi_n(t)\cdot s^{\frac{n-1}{2}}\exp\left(-\frac{\sqrt{r^2-t^2}}{\sqrt{a}}\sqrt{s}\right)dt-\\&-\int_r^R \frac{1}{\sqrt{t^2-r^2}}\frac{1}{\sqrt{s}}\exp\left(-\frac{2R}{\sqrt{a}}\sqrt{s}\right)\sum_{n=0}^{\infty}\varphi_n(t)s^{\frac{n-1}{2}}\sin\left(\frac{\sqrt{t^2-r^2}}{\sqrt{a}}\sqrt{s}\right)dt=\overline{f}(r,s)-\frac{T_0}{s}.\end{aligned} \tag{18}$$

It is possible to find new formulas of inverse integral Laplace transforms:

$$L^{-1}\left[\frac{1}{\sqrt{s}}\exp\left(-\frac{R}{\sqrt{a}}\sqrt{s}\right)\exp\left(-\frac{\sqrt{r^2-t^2}}{\sqrt{a}}\sqrt{s}\right)\right]=\frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{\left(R+\sqrt{r^2-t^2}\right)^2}{4a\tau}\right), \tag{19}$$

$$L^{-1}\left[\frac{1}{\sqrt{s}}\exp\left(-\frac{R}{\sqrt{a}}\sqrt{s}\right)\sin\left(\frac{\sqrt{t^2-r^2}}{\sqrt{a}}\sqrt{s}\right)\right]=\frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{R^2-\left(t^2-r^2\right)}{4a\tau}\right)\sin\left(\frac{R\sqrt{t^2-r^2}}{2a\tau}\right), \tag{20}$$

$$L^{-1}\left[\exp\left(-\frac{R}{\sqrt{a}}\sqrt{s}\right)s^{\frac{n-1}{2}}\right]=\frac{1}{2^n\sqrt{\pi\tau^{n+1}}}\exp\left(-\frac{R^2}{4a\tau}\right)\mathrm{H_n}\left(\frac{R}{2\sqrt{a\tau}}\right),\ n=0,1,2,\dots, \tag{21}$$

where $\mathrm{H_n}(x)$ are orthogonal Hermit polynomials.

By the using of inverse integral Laplace transforms (19), (20), (21), $L^{-1}\left[\overline{f}(r,s)-\frac{T_0}{s}\right]=f(r,\tau)-T_0$ and $L^{-1}\left[\frac{1}{\sqrt{s}}\right]=\frac{1}{\sqrt{\pi\tau}}$ we can obtain formula

$$L^{-1}\left[\sum_{n=0}^{\infty}\frac{1}{\sqrt{s}}\varphi_n(t)\exp\left(-\frac{2R}{\sqrt{a}}\sqrt{s}\right)s^{\frac{n-1}{2}}\right]=\sum_{n=0}^{\infty}\varphi_n(t)\int_0^\tau \frac{1}{\sqrt{\pi(\tau-\xi)}}\frac{\exp\left(-\frac{R^2}{a\xi}\right)}{2^n\sqrt{\pi\cdot\xi^{n+1}}}\cdot\mathrm{H_n}\left(\frac{R}{\sqrt{a\xi}}\right)d\xi.$$

The equation (18) reduces in the form

$$\int_0^r \frac{dt}{\sqrt{r^2-t^2}} \sum_{n=0}^{\infty} \varphi_n(t) \int_0^r \frac{1}{2^n \sqrt{\pi \xi^{n+1}} \sqrt{\pi(\tau-\xi)}} \exp\left(-\frac{\left(R+\sqrt{r^2-t^2}\right)^2}{4a(\tau-\xi)}\right) \Omega_n\left(\frac{R}{2\sqrt{a\xi}}\right) d\xi -$$

$$-\int_r^R \frac{dt}{\sqrt{t^2-r^2}} \sum_{n=0}^{\infty} \varphi_n(t) \int_0^r \frac{1}{2^n \sqrt{\pi \xi^{n+1}} \sqrt{\pi(\tau-\xi)}} \exp\left(-\frac{\left(R+\sqrt{t^2-r^2}\right)^2}{4a(\tau-\xi)}\right) \Omega_n\left(\frac{R}{2\sqrt{a\xi}}\right) \cdot \tag{22}$$

$$\cdot \sin\left(\frac{R\sqrt{t^2-r^2}}{2a(\tau-\xi)}\right) d\xi = f(r,\tau) - T_0,$$

where $\Omega_n(x) = \exp\left(-x^2\right) \mathrm{H_n}(x)$, $0 < r < R$, $\tau > 0$.

It is known, that a quasi-stationary phase of heat transfer process of the semi-bounded isotropic solid with mixed boundary conditions (8) and (9) will occur at the certain moment $\tau > 0$ (or, more exactly, when a value of the Fourier criterion $Fo = a\tau / R^2$ will be more than 1).

At $Fo < 1$ a process of heating of a considering solid is very complicated and the great many of terms of a series (17) also is necessary to calculate with a sufficient accuracy. And, factors $\varphi_n(r)$ should be determinate at identical complex values of L-parameter in the left and right parts of an integral equation (16). In a quasi-stationary heating phase the given development of temperature fields on a surface of a half-space may be investigated with the certain accuracy already at the account only the first terms of the appropriate infinite series.

Thus, in case of a quasi-stationary phase a value of factor $\varphi_0(r)$ can be possible to determine from the appropriate integral equation obtained from (22) by the $n = 0$, by the replacement of a variable $x = R / \left(2\sqrt{a\xi}\right)$, by the using of known formula [4, p.344]

$$\int_0^{\infty} \frac{1}{\sqrt{x}(x+y)} \exp\left(-px - \frac{q}{x}\right) dx = \frac{\pi}{\sqrt{y}} \exp\left(py + \frac{q}{y}\right) \mathrm{erfc}\left(\sqrt{\frac{q}{y}} + \sqrt{py}\right)$$

and by the calculation of corresponding integrals we come to the following integral equation:

$$\int_0^r \frac{\varphi_0(t)}{\sqrt{r^2-t^2}} \mathrm{erfc}\left(\frac{2R+\sqrt{r^2-t^2}}{2\sqrt{a\tau}}\right) dt - \frac{1}{2i} \int_r^R \frac{\varphi_0(t)}{\sqrt{t^2-r^2}} \left[\mathrm{erfc}\left(\frac{2R - i\sqrt{t^2-r^2}}{2\sqrt{a\tau}}\right) - \right.$$

$$\left. - \mathrm{erfc}\left(\frac{2R + i\sqrt{t^2-r^2}}{2\sqrt{a\tau}}\right)\right] dt = f(r,\tau) - T_0, \quad 0 < r < R, \quad Fo > 1, \tag{23}$$

where at small values of argument x the approximate equality $\mathrm{erfc}(x) \cong 1 - 2x / \sqrt{\pi}$ is valid.

Hence, in a quasi-stationary phase with the using of the notation $F(r,\tau) = f(r,\tau) - T_0$ it is possible to write a solution of the equation (23) as

$$\varphi_0(r) = \frac{2\sqrt{\pi a\tau}}{\pi\left(\sqrt{\pi a\tau} - 2R\right)} \frac{d}{dr} \int_0^r F(\mu,\tau) \frac{\mu\, d\mu}{\sqrt{r^2-\mu^2}} +$$

$$+ \frac{4\sqrt{\pi a\tau}}{\pi^2 \left[\sqrt{\pi a\tau} - \left(1 + \frac{1}{\pi}\right) 2R\right] \left(\sqrt{\pi a\tau} - 2R\right)} \int_0^R F(\mu,\tau) \frac{\mu}{\sqrt{R^2-\mu^2}} d\mu, \quad 0 < r < R, \quad Fo > 1. \tag{24}$$

Note here, that in stationary phase ($\tau = \infty$) from formula (24) we can be obtain known formula

$$\varphi_0^*(r) = \frac{2}{\pi} \frac{d}{dr} \int_0^r f^*(\mu) \frac{\mu\, d\mu}{\sqrt{r^2-\mu^2}}, \quad 0 < r < R, \quad Fo > 1.$$

where $f^{*}(r) = \lim_{\tau \to \infty} f(r, \tau)$ is the given distribution of absolute stationary temperature in circular area $0 < r < R$ on a surface of a half-space.

In conclusion let’s mark the possibility of research of a quasi-stationary heat transfer phase on model of the isotropic half-space with mixed boundary conditions on its surface. For this purpose we can write the analytical solution of mathematical problem (1) – (6) by using formulas (12), (15), (17) at $n = 0$ and (24).

References

[1]. V.P. Kozlov, P.A. Mandrik, “Systems of Integral and Differential Equations with L-parameter in Problems of Mathematical Physics and Identification Methods of Heat Characteristics”, Minsk, 2000. (In Russian).

[2]. P.A. Mandrik, “Applications of Laplace and Hankel transforms to solution of mixed nonstationary boundary value problems”, *Integral Transforms Spec. Funct.*, **13**, 3, (2002), 277-283.

[3]. V.P. Kozlov, P.A. Mandrik, N.I. Yurchuk, “The method of dual integral equations into L-image range for solution of two-dimensional nonstationary heat transfer problems under mixed boundary conditions”, *Jour. Eng. Phys. and Thermophys.*, **71**, 4, (1998), 734-743. (In Russian).

[4]. A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, “Integrals and Series. Elementary functions”, Moscow, 1981. (In Rusian).