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NUMERICAL METHODS

A Criterion for Coefficient Stability

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INTRODUCTION

In the analysis of initial-boundary value problems for nonstationary equations of mathematical physics, attention is paid mainly to the stability of the solution with respect to the initial data and the right-hand side. It was proved that the stability of a two-level operator-difference scheme with respect to the initial data is necessary and sufficient for its stability with respect to the right-hand side [1, pp. 95–97].

However, when solving a differential problem, one can face a situation in which the coefficients of the equation are known approximately rather than exactly. (For example, they are obtained with the use of some numerical algorithm, as a result of physical measurements, etc.) It is therefore important to analyze the stability of the solution of the differential problem under perturbations of the initial conditions, the right-hand side, and the operator coefficients (strong stability). Similar problems arise for finite-difference approximations to the differential problem.

The first results concerning the strong stability analysis of operator-difference schemes approximating nonstationary problems of mathematical physics were given in [2, 3].

In 1999, Gulin conjectured that stability with respect to the initial data should imply not only stability with respect to the right-hand side but also coefficient stability; this assumption was proved in [4]. Later, Makarov put forward the conjecture that the three notions (stability with respect to the initial data, stability with respect to the right-hand side, and coefficient stability) are equivalent.

In Section 1 of the present paper, by analogy with [5], we give the definition of strong stability of a two-level operator-difference scheme. In Section 2, we introduce the notions of stability of a scheme with respect to the initial data, stability with respect to the right-hand side, and coefficient stability. If only the initial data and the right-hand side are perturbed in the original scheme, then stability with respect to the right-hand side and the initial data, together with the triangle inequality, implies the stability of the difference scheme. However, if the operator coefficients are also perturbed, then the perturbation problem becomes nonlinear and stability with respect to the initial data and the right-hand side and coefficient stability do not imply the strong stability of the difference scheme. In Section 3, we prove the above-mentioned Gulin–Makarov conjecture.

1. STRONG STABILITY OF TWO-LEVEL OPERATOR-DIFFERENCE SCHEMES

Let H_h be a real finite-dimensional space whose dimension depends on h and can tend to infinity as $|h| \to 0$. Here h is a vector parameter equipped with a norm $|h| > 0$.

Let $\hat{\bar{\omega}}_\tau = \{t_n = t_{n-1} + \tau_n,\ n = 0, \dots, n_0;\ t_0 = 0,\ t_{n_0} = T\} = \hat{\omega}_\tau \cup \{T\}$ be an arbitrary grid on the interval $0 \le t \le T$ with increments $\tau_n = t_n - t_{n-1}$.

Consider the Cauchy problem for the two-level operator-difference scheme

$$By_t + Ay = \varphi, \qquad t \in \hat{\omega}_\tau, \qquad y(0) = u_0, \tag{1.1}$$

where $A = A_{h\tau}(t_n)$ and $B = B_{h\tau}(t_n) : H_h \to H_h$ are linear operators, in general, depending on τ and t_n, $y = y_n = y(t_n) \in H_h$ is the unknown function, and $\varphi = \varphi_n = \varphi(t_n)$ and $u_0 \in H_h$ are given. The operators $A_{h\tau}(t_n)$ and $B_{h\tau}(t_n)$ are bounded for any given h and τ but, in general, not uniformly bounded with respect to h and τ.

We use the index-free notation of the theory of difference schemes [1, 6]:

$$y = y_n = y(t_n), \qquad \hat{y} = y_{n+1} = y(t_{n+1}), \qquad y_t = y_{t,n} = (y_{n+1} - y_n)/\tau_{n+1}.$$

Along with problem (1.1), we consider the perturbed problem

$$\tilde{B}\tilde{y}_t + \tilde{A}\tilde{y} = \tilde{\varphi}, \qquad t \in \hat{\omega}_\tau, \qquad \tilde{y}(0) = \tilde{u}_0. \tag{1.2}$$

Let us proceed to the study of the strong stability of difference schemes. A solution of the difference Cauchy problem (1.1) is an abstract function $y_{h\tau}(t_n)$ depending on the discrete argument $t_n \in \hat{\omega}_\tau$ and ranging in H_h. The input data of the problem consists of the initial vector $y_0 = y_{0,h\tau} \in H_h$ and the right-hand side $\varphi = \varphi_{h\tau}(t_n)$, which is a given abstract function of the discrete argument $t_n \in \hat{\omega}_\tau$ and ranges in H_h. Suppose that H_h is a normed space, or, more precisely, is equipped with some norms $\|y_n\|_{(1_n)}$ and $\|\varphi_n\|_{(2_n)}$ in which we estimate the perturbation of the solution and the right-hand side of Eq. (1.1), respectively. These norms can depend on $t = t_n$, h, and $\{\tau_k\}$.

We introduce the perturbation $\delta = \tilde{y} - y$ of the solution of problem (1.1) with respect to the solution of problem (1.2); it satisfies the problem

$$\tilde{B}\delta_t + \tilde{A}\delta = (\tilde{\varphi} - \varphi) + \left(B - \tilde{B}\right) y_t + \left(A - \tilde{A}\right) y, \qquad t \in \hat{\omega}_\tau, \qquad \delta_0 = \tilde{u}_0 - u_0. \tag{1.3}$$

If the operators A_n and $\tilde{A}_n$ are treated as mappings $A_n, \tilde{A}_n : H_h^{(1_n)} \to H_h^{(2_n)}$ and the operators B_n and $\tilde{B}_n$ as mappings $B_n, \tilde{B}_n : H_h^{(3_n)} \to H_h^{(4_n)}$, where $H_h^{(\alpha)} \subset H_h$ $(\alpha = 1_n, 3_n)$ and $H_h \subset H_h^{(\alpha)}$ $(\alpha = 2_n, 4_n)$, then it is natural to estimate the perturbations of the operator coefficients A_n and B_n in the operator norms

$$\left\|A_n - \tilde{A}_n\right\|_{(3_n)} = \left\|A_n - \tilde{A}_n\right\|_{H_h^{(1_n)} \to H_h^{(2_n)}} = \sup_{\|u_n\|_{H_h^{(1_n)}} \neq 0} \frac{\left\|\left(A_n - \tilde{A}_n\right) u_n\right\|_{H_h^{(2_n)}}}{\|u_n\|_{H_h^{(1_n)}}},$$

$$\left\|B_n - \tilde{B}_n\right\|_{(4_n)} = \left\|B_n - \tilde{B}_n\right\|_{H_h^{(3_n)} \to H_h^{(4_n)}} = \sup_{\|u_n\|_{H_h^{(3_n)}} \neq 0} \frac{\left\|\left(B_n - \tilde{B}_n\right) u_n\right\|_{H_h^{(4_n)}}}{\|\tilde{u}_n\|_{H_h^{(3_n)}}}.$$

Following [5], we introduce the notion of strong stability.

Definition 1.1. The operator-difference scheme (1.1) is said to be *strongly stable* if it is stable under perturbations of the input data, viz., the initial conditions, the right-hand side, and the operator coefficients. In other words, there exist positive constants M_k, $k = 1, 2, 3, 4$, such that the *a priori* estimate

$$\begin{aligned} \|\tilde{y}_n - y_n\|_{(1_n)} \le M_1 \|\tilde{u}_0 - u_0\|_{(1_0)} &+ M_2 \sum_{k=0}^{n-1} \tau_{k+1} \|\tilde{\varphi}_k - \varphi_k\|_{(2_k)} \\ &+ M_3 \sum_{k=0}^{n-1} \tau_{k+1} \left\|A_k - \tilde{A}_k\right\|_{(3_k)} + M_4 \sum_{k=0}^{n-1} \tau_{k+1} \left\|B_k - \tilde{B}_k\right\|_{(4_k)} \end{aligned} \tag{1.4}$$

is valid.

Here and throughout the following, we assume that $0 \le n \le n_0 < \infty$.

Example 1.1. The main problem in the derivation of estimates of the form (1.4) is to choose the spaces in which the operators A_n, $\tilde{A}_n$ and B_n, $\tilde{B}_n$ act so as to ensure that the norms of perturbations of the operator coefficients are bounded.

By way of example, we consider the constant operator $Ay = -(ay_{\bar{x}})_x$, $x \in \omega_h$, $y_0 = y_N = 0$, where ω_h is the uniform grid with increment $h = 1/N$ on $[0, 1]$ and $a \neq a(t) \geq \delta > 0$ and $y = y(t)$ are grid functions. Then $\tilde{A}y = -(\tilde{a}y_{\bar{x}})_x$, $x \in \omega_h$, $y_0 = y_N = 0$, $\tilde{a} \neq \tilde{a}(t) \geq \delta > 0$, and $\left(A - \tilde{A}\right) y = -((a - \tilde{a})\, y_{\bar{x}})_x$, $x \in \omega_h$, $y_0 = y_N = 0$. The space H_h is defined as the set of grid functions $y_i = y(x_i)$ defined on ω_h and vanishing for $i = 0, N$. The inner product and the norm in H_h are given by the formulas

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h, \qquad \|y\| = \sqrt{(y, y)}.$$

As $H_h^{(1_n)}$ we take, say, the normed space $H_{\tilde{A}^*\tilde{A}}$ with the norm $\|\tilde{A}u\|$, $u \in H_h$, and $H_h^{(2_n)} = H_h$. If $|\tilde{a}(x) - a(x)| \leq \alpha_1 < +\infty$ and $|\tilde{a}_x(x) - a_x(x)| \leq \alpha_2 < +\infty$, then

$$\left\|\left(A - \tilde{A}\right) u\right\|_{H_h} \leq \alpha \left\|\tilde{A}u\right\|_{H_h} = \alpha \|u\|_{H_{\tilde{A}^*\tilde{A}}};$$

consequently, for the perturbation of the operator A, we have

$$\left\|A - \tilde{A}\right\|_{H_{\tilde{A}^*\tilde{A}} \to H_h} \leq \alpha < +\infty.$$

2. STABILITY WITH RESPECT TO THE INPUT DATA

The notion of stability with respect to the initial data and the right-hand side, as well as the notion of coefficient stability for a stationary problem, was introduced in [6] for the difference scheme (1.1).

Here we introduce the corresponding notions for the case of a nonstationary problem and perturbations of operator coefficients.

Along with problem (1.2), we consider the problems

$$B\tilde{y}_t^{(1)} + A\tilde{y}^{(1)} = \varphi, \qquad t \in \hat{\omega}_\tau, \qquad \tilde{y}^{(1)}(0) = \tilde{u}_0, \tag{1.2a}$$
$$B\tilde{y}_t^{(2)} + A\tilde{y}^{(2)} = \tilde{\varphi}, \qquad t \in \hat{\omega}_\tau, \qquad \tilde{y}^{(2)}(0) = u_0, \tag{1.2b}$$
$$\tilde{B}\tilde{y}_t^{(3)} + \tilde{A}\tilde{y}^{(3)} = \varphi, \qquad t \in \hat{\omega}_\tau, \qquad \tilde{y}^{(3)}(0) = u_0. \tag{1.2c}$$

Definition 2.1. The operator-difference scheme (1.1) is said to be *stable with respect to the initial data* if there exists a positive constant M_1 such that

$$\left\|\tilde{y}_n^{(1)} - y_n\right\|_{(1_n)} \leq M_1 \left\|\tilde{u}_0 - u_0\right\|_{(1_0)}. \tag{2.1}$$

Definition 2.2. The operator-difference scheme (1.1) is said to be *stable with respect to the right-hand side* if there exists a positive constant M_2 such that

$$\left\|\tilde{y}_n^{(2)} - y_n\right\|_{(1_n)} \leq M_2 \sum_{k=1}^{n-1} \tau_{k+1} \left\|\tilde{\varphi}_k - \varphi_k\right\|_{(2_k)}. \tag{2.2}$$

Definition 2.3. The operator-difference scheme (1.1) is *stable with respect to the operator coefficients* if there exist positive constants M_3 and M_4 such that

$$\left\|\tilde{y}_n^{(3)} - y_n\right\|_{(1_n)} \leq M_3 \sum_{k=1}^{n-1} \tau_{k+1} \left\|\tilde{A}_k - A_k\right\|_{(3_k)} + M_4 \sum_{k=1}^{n-1} \tau_{k+1} \left\|\tilde{B}_k - B_k\right\|_{(4_k)}. \tag{2.3}$$

Furthermore, we introduce the perturbations of the solution of problem (1.1) with respect to the solutions of problems (1.2a)–(1.2c): $\delta^{(k)} = \tilde{y}^{(k)} - y$, $k = 1, 2, 3$. For $\delta^{(k)}$ $(k = 1, 2, 3)$, we obtain the problems

$$B\delta_t^{(1)} + A\delta^{(1)} = 0, \qquad t \in \hat{\omega}_\tau, \qquad \delta_0^{(1)} = \tilde{u}_0 - u_0, \tag{2.4}$$

$$B\delta_t^{(2)} + A\delta^{(2)} = \tilde{\varphi} - \varphi, \qquad t \in \hat{\omega}_\tau, \qquad \delta_0^{(2)} = 0, \tag{2.5}$$

$$B\delta_t^{(3)} + A\delta^{(3)} = \left(B - \tilde{B}\right)\tilde{y}_t^{(3)} + \left(A - \tilde{A}\right)\tilde{y}^{(3)}, \qquad t \in \hat{\omega}_\tau, \qquad \delta_0^{(3)} = 0. \tag{2.6}$$

Moreover, $\delta^{(3)}$ is a solution of the problem

$$\tilde{B}\delta_t^{(3)} + \tilde{A}\delta^{(3)} = \left(B - \tilde{B}\right)y_t + \left(A - \tilde{A}\right)y, \qquad t \in \hat{\omega}_\tau, \qquad \delta_0^{(3)} = 0.$$

Note that since problem (1.3) is nonlinear in the case of perturbation of the operator coefficients, and since $\delta \neq \delta^{(1)} + \delta^{(2)} + \delta^{(3)}$, it follows that, unlike the case of perturbations of the initial data and the right-hand side alone, it is impossible to use the triangle inequality to obtain a strong stability estimate for δ from the estimates for $\delta^{(1)}$, $\delta^{(2)}$, and $\delta^{(3)}$.

We assume that the Cauchy problem (1.1) is solvable, i.e., the inverse operator B_n^{-1} exists. Since the operator $B_n = B_{h\tau}(t_n)$ is bounded for given h and τ_n, i.e.,

$$\|B_n u_n\| \le m\,\|u_n\| \qquad (m > 0),$$

it follows that the operator B_n^{-1} satisfies $\|B_n^{-1}u_n\| \ge (1/m)\,\|u_n\|$ for given h and τ_n; consequently, the expression $\|B_n^{-1}u_n\|$ is a norm. The scheme (1.1) can be represented in the form [1]

$$y_{n+1} = S_{n+1}y_n + \tau_{n+1}B_n^{-1}\varphi_n, \qquad n = 0, 1, \ldots, \qquad y_0 \in H, \tag{2.7}$$

where the operator S_{n+1} of transition from level n to level $n+1$ is equal to $S_{n+1} = E - \tau_{n+1}B_n^{-1}A_n$.

By successively using formula (2.7), we obtain

$$y_n = T_{n,0}y_0 + \sum_{k=0}^{n-1} \tau_{k+1}T_{n,k+1}B_k^{-1}\varphi_k. \tag{2.8}$$

Furthermore,

$$y_n = T_{n,k}y_k + \sum_{j=k}^{n-1} \tau_{j+1}T_{n,j+1}B_j^{-1}\varphi_j$$

for each $k \le n - 1$. Here $T_{n,k}$ is the operator of transition from level k to level n:

$$T_{n,k} = S_n S_{n-1} \cdots S_{k+1}, \qquad T_{n,n} = E,$$

and $T_{n,0}$ is the resolving operator.

By analogy with [1], we say that the scheme (1.1) is *uniformly stable with respect to the initial data* if there exists a positive constant M_1 independent of h, $\{\tau_k\}$, and the choice of the initial data such that

$$\left\|\tilde{y}_n^{(1)} - y_n\right\|_{(1_n)} \le M_1 \left\|\tilde{y}_k^{(1)} - y_k\right\|_{(1_k)}, \qquad k = 0, \ldots, n-1, \qquad n = 1, 2, \ldots \tag{2.9}$$

Obviously, the estimate (2.9) is valid if and only if the operator $T_{n,k}$ of transition from level k to level n is bounded uniformly with respect to n and k:

$$\|T_{n,k}\| \le M_1, \qquad 0 \le k \le n-1, \qquad n = 1, 2, \ldots \tag{2.10}$$

3. THE RELATIONSHIP BETWEEN STABILITY WITH RESPECT TO THE INITIAL DATA, STABILITY WITH RESPECT TO THE RIGHT-HAND SIDE, AND COEFFICIENT STABILITY

Theorem 3.1. Suppose that the operators B_n, $\tilde{B}_n$, and $\tilde{A}_n$ are continuously invertible for all $1 \le n \le n_0 < +\infty$, the operators $\tilde{A}_n$, $\tilde{A}_k$, and $\tilde{B}_k$ commute for all $0 \le k, n \le n_0$, the operators S_1 and $\tilde{S}_1$ are bounded in the norm $\|\cdot\|_{H_h \to H_h}$, i.e.,

$$\|S_1\| = \|S_1\|_{H_h \to H_h} \le M, \qquad \left\|\tilde{S}_1\right\| = \left\|\tilde{S}_1\right\|_{H_h \to H_h} \le \tilde{M}, \tag{3.1}$$

and the operator $\tilde{A}_n$ satisfies the inequality

$$\left\|\left(\tilde{A}_{n+1} - \tilde{A}_n\right) u_n\right\|_{H_h} \le c_0 \tau_{n+1} \left\|\tilde{A}_n u_n\right\|_{H_h}. \tag{3.2}$$

Then the following assertions are equivalent:
(1) the scheme (1.1) is uniformly stable with respect to the initial data;
(2) the scheme (1.1) is stable with respect to the right-hand side;
(3) the scheme (1.1) is coefficient stable.

Proof. (1) ⇒ (2). Suppose that the scheme (1.1) is uniformly stable with respect to the initial data, i.e., an estimate of the form (2.9) is valid for problem (2.4):

$$\left\|\delta_n^{(1)}\right\|_{(1_n)} \le M_1 \left\|\delta_k^{(1)}\right\|_{(1_k)}, \qquad k = 0, \dots, n-1, \qquad n = 1, 2, \dots \tag{3.3}$$

Then inequality (2.10) is also valid with $\|T_{n,k}\| = \|T_{n,k}\|_{H_h \to H_h}$.

By virtue of (2.8), the solution of problem (2.5) is given by the formula

$$\delta_n^{(2)} = \sum_{k=0}^{n-1} \tau_{k+1} T_{n,k+1} B_k^{-1} \left(\tilde{\varphi}_k - \varphi_k\right). \tag{3.4}$$

This, together with (2.10), implies that

$$\begin{aligned} \left\|\delta_n^{(2)}\right\|_{(1_n)} &\le \sum_{k=0}^{n-1} \tau_{k+1} \left\|T_{n,k+1}\right\| \left\|B_k^{-1}\left(\tilde{\varphi}_k - \varphi_k\right)\right\|_{(1_{k+1})} \\ &\le M_1 \sum_{k=0}^{n-1} \tau_{k+1} \left\|B_k^{-1}\left(\tilde{\varphi}_k - \varphi_k\right)\right\|_{(1_{k+1})} \\ &\le M_1 \sum_{k=0}^{n-1} \tau_{k+1} \left\|\tilde{\varphi}_k - \varphi_k\right\|_{(2_k)}. \end{aligned} \tag{3.5}$$

Therefore, the scheme (1.1) is stable with respect to the right-hand side under the norm compatibility condition

$$\|v_k\|_{(2_k)} = \left\|B_k^{-1} v_k\right\|_{(1_{k+1})}.$$

(2) ⇒ (3). Suppose that the scheme (1.1) is stable with respect to the right-hand side, i.e., the estimate (3.5) be valid. We shall prove the boundedness of the operators $T_{n,k}$ by the technique in [1]. We choose the perturbed problem (1.2b) so as to ensure that $\tau_{k+1} B_k^{-1} \left(\tilde{\varphi}_k - \varphi_k\right) = \delta_{k,k_0} \eta$, where δ_{k,k_0} is the Kronecker delta. Then from (3.5), we have

$$\left\|\delta_n^{(2)}\right\|_{(1_n)} \le M_1 \|\eta\|_{(1_{k_0+1})}, \qquad n = 1, \dots, n_0, \qquad k_0 = 0, \dots, n-1.$$

On the other hand, from (3.4), we obtain

$$\delta_n^{(2)} = T_{n,k_0+1}\eta, \qquad \left\|\delta_n^{(2)}\right\|_{(1_n)} = \|T_{n,k_0+1}\eta\|_{(1_n)} \le M_1\|\eta\|_{(1_{k_0+1})},$$

and consequently, $\|T_{n,k}\| \le M_1$ for all $k = 1, \ldots, n-1$. Since $T_{n,0} = T_{n,1}S_1$, it follows from (3.1) that $\|T_{n,0}\| \le \|T_{n,1}\|\|S_1\| \le M_1 M$, i.e., the norms of $T_{n,k}$ are bounded for all $k = 0, \ldots, n$ and all $n = 1, \ldots, n_0$.

Let us estimate the norms of the operators $\tilde{T}_{n,k}$ and $\tilde{S}_k = E + \tau_{k+1}\tilde{B}_k^{-1}\tilde{A}_k^{-1}$. To this end, we consider the quantity $\nu_k = \delta_k^{(1)} + \delta_k^{(2)} + \delta_k^{(3)}$. One can readily see that ν is a solution of the problem

$$\tilde{B}\nu_t + \tilde{A}\nu = \tilde{B}\left(\delta^{(1)} + \delta^{(2)} - y\right)_t + \tilde{A}\left(\delta^{(1)} + \delta^{(2)} - y\right) + \varphi, \quad t \in \hat{\omega}_\tau, \qquad \nu_0 = \tilde{u}_0 - u_0 = \delta_0,$$

and the representation

$$\nu_n = \tilde{T}_n\delta_0 + \sum_{k=0}^{n-1}\tau_{k+1}\tilde{T}_{n,k+1}\tilde{B}_k^{-1}\left(\tilde{B}_k\left(\delta^{(1)} + \delta^{(2)} - y\right)_{t,k} + \tilde{A}_k\left(\delta^{(1)} + \delta^{(2)} - y\right)_k + \varphi_k\right)$$

is valid. Since the solution of problem (1.3) can be represented in the form

$$\delta_n = \tilde{T}_n\delta_0 + \sum_{k=0}^{n-1}\tau_{k+1}\tilde{T}_{n,k+1}\tilde{B}_k^{-1}\left(\tilde{\varphi}_k - \tilde{B}_k y_{t,k} - \tilde{A}_k y_k\right),$$

we have

$$\begin{aligned}\delta_n &= \delta_n^{(1)} + \delta_n^{(2)} + \delta_n^{(3)} \\ &\quad + \sum_{k=0}^{n-1}\tau_{k+1}\tilde{T}_{n,k+1}\tilde{B}_k^{-1}\left(\tilde{\varphi}_k - \varphi_k - \tilde{B}_k\left(\delta^{(1)} + \delta^{(2)}\right)_{t,k} - \tilde{A}_k\left(\delta^{(1)} + \delta^{(2)}\right)_k\right).\end{aligned}$$

From the last relation, for $\delta^{(2)}$, we obtain

$$\begin{aligned}\delta_n^{(2)} &= \left(\tilde{T}_n - T_n\right)\delta_0 + \sum_{k=0}^{n-1}\tau_{k+1}\tilde{T}_{n,k+1}\tilde{B}_k^{-1} \\ &\quad \times \left(\tilde{B}_k\left(\delta^{(1)} + \delta^{(2)} + y\right)_{t,k} + \tilde{A}_k\left(\delta^{(1)} + \delta^{(2)} + y\right)_k - \tilde{\varphi}_k\right).\end{aligned}$$

We choose the initial data $\tilde{u}_0$ and u_0 and the right-hand sides $\tilde{\varphi}_k$ and φ_k as follows:

$$\begin{aligned}&\tilde{u}_0 = u_0, \\ &\varphi_k = \tilde{B}_k\left(\delta^{(1)} + \delta^{(2)} + y\right)_{t,k} + \tilde{A}_k\left(\delta^{(1)} + \delta^{(2)} + y\right)_k - \frac{1}{\tau_{k+1}}\delta_{k,k_0}\left(\tilde{B}_k + \tilde{B}_k\right)\eta, \\ &\tilde{\varphi}_k = \varphi_k + \frac{1}{\tau_{k+1}}\delta_{k,k_0}B_k\eta.\end{aligned}$$

Then $\delta_n^{(2)} = \tilde{T}_{n,k_0+1}\eta$, $\tau_{k+1}B_k^{-1}(\tilde{\varphi}_k - \varphi_k) = \delta_{k,k_0}\eta$, and it follows from the estimate (3.5) that $\|\tilde{T}_{n,k_0+1}\eta\|_{(1_n)} \le M_1\|\eta\|_{(1_{k_0+1})}$. Since k_0 and n are arbitrary, we have the estimate

$$\left\|\tilde{T}_{n,k}\right\| \le M_1 \quad \text{for all} \quad 1 \le k \le n, \quad n = 1, 2, \ldots$$

The formula $\tilde{T}_{n,0} = \tilde{T}_{n,1}\tilde{S}_1$ implies the estimate

$$\left\|\tilde{T}_{n,0}\right\| \le \left\|\tilde{T}_{n,1}\right\|\left\|\tilde{S}_1\right\| \le M_1M_3. \tag{3.6}$$

Consequently, the norms of $\tilde{T}_{n,k}$ are bounded for all $0 \le k < n$ and $n = 1, \ldots, n_0$.

By virtue of (2.8), the solution of problem (2.6) is given by the formula

$$\delta_n^{(3)} = \sum_{k=0}^{n-1} \tau_{k+1} T_{n,k+1} B_k^{-1} \left(\left(B_k - \tilde{B}_k \right) \tilde{y}_{t,k}^{(3)} + \left(A_k - \tilde{A}_k \right) \tilde{y}_k^{(3)} \right).$$

Hence we have

$$\begin{aligned} \left\| \delta_n^{(3)} \right\|_{(1_n)} &\le \sum_{k=0}^{n-1} \tau_{k+1} \left\| T_{n,k+1} \right\| \left(\left\| B_k^{-1} \left(B_k - \tilde{B}_k \right) \tilde{y}_{t,k}^{(3)} \right\|_{(1_k)} + \left\| B_k^{-1} \left(A_k - \tilde{A}_k \right) \tilde{y}_k^{(3)} \right\|_{(1_k)} \right) \\ &\le M_1 \sum_{k=0}^{n-1} \tau_{k+1} \left(\left\| A_k - \tilde{A}_k \right\|_{(3_k)} \left\| \tilde{A}_k \tilde{y}_k^{(3)} \right\|_{(1_k)} + \left\| B_k - \tilde{B}_k \right\|_{(4_k)} \left\| \tilde{B}_k \tilde{y}_{t,k}^{(3)} \right\|_{(1_k)} \right), \end{aligned}$$

where

$$\begin{aligned} \left\| A_k - \tilde{A}_k \right\|_{(3_k)} &= \left\| A_k - \tilde{A}_k \right\|_{H_{\tilde{A}_k^* \tilde{A}_k} \to H_{B_k^{*-1} B_k^{-1}}} \\ &= \sup_{\| \tilde{A}_k u_k \|_{(1_k)} \neq 0} \left(\left\| B_k^{-1} \left(A_k - \tilde{A}_k \right) u_k \right\|_{(1_k)} \Big/ \left\| \tilde{A}_k u_k \right\|_{(1_k)} \right), \\ \left\| B_k - \tilde{B}_k \right\|_{(4_k)} &= \left\| B_k - \tilde{B}_k \right\|_{H_{\tilde{B}_k^* \tilde{B}_k} \to H_{B_k^{*-1} B_k^{-1}}} \\ &= \sup_{\| \tilde{B}_k u_k \|_{(1_k)} \neq 0} \left(\left\| B_k^{-1} \left(B_k - \tilde{B}_k \right) u_k \right\|_{(1_k)} \Big/ \left\| \tilde{B}_k u_k \right\|_{(1_k)} \right). \end{aligned}$$

From Eq. (1.2c), we obtain $\left\| \tilde{B}_k \tilde{y}_{t,k} \right\|_{(1_k)} \le \| \varphi_k \|_{(1_k)} + \left\| \tilde{A}_k \tilde{y}_k^{(3)} \right\|_{(1_k)}$.

Let us estimate $\left\| \tilde{A}_k \tilde{y}_k^{(3)} \right\|_{(1_k)}$. The solution of problem (1.2c) is given by the formula

$$\tilde{y}_n^{(3)} = \tilde{T}_n u_0 + \sum_{k=0}^{n-1} \tau_{k+1} \tilde{T}_{n,k+1} \tilde{B}_k^{-1} \varphi_k.$$

Then, by taking account of the estimate (3.6), the continuous invertibility of the operator $\tilde{B}_k$, and the fact that

$$\left\| \tilde{A}_k v_k \right\|_{(1_k)} \le \varrho \left\| \tilde{A}_{k-1} v_k \right\|_{(1_{k-1})}, \qquad \varrho = \max_{1 \le k \le n} \varrho_k, \qquad \varrho_k = 1 + C_0 \tau_k,$$

which follows from condition (3.2), we obtain the inequality

$$\left\| \tilde{A}_k \tilde{y}_k^{(3)} \right\| \le M_1 \left(\varrho^k \left\| \tilde{A}_0 u_0 \right\|_{(1_0)} + \sum_{j=0}^{k-1} \tau_{j+1} \varrho^{k-j} \left\| \tilde{A}_j \tilde{B}_j^{-1} \varphi_j \right\|_{(1_j)} \right).$$

With regard to the last estimate, we obtain the inequality

$$\left\| \delta_n^{(3)} \right\|_{(1_n)} \le M_3 \sum_{k=0}^{n-1} \tau_{k+1} \left\| A_k - \tilde{A}_k \right\|_{(3_k)} + M_4 \sum_{k=0}^{n-1} \tau_{k+1} \left\| B_k - \tilde{B}_k \right\|_{(4_k)},$$

where

$$M_i \le M_1 \left(\varrho^n \left\| \tilde{A}_0 u_0 \right\| + (i-3) \max_{0 \le k \le n-1} \| \varphi_k \|_{(1_k)} + \sum_{j=0}^{n-1} \tau_{j+1} \varrho^{n-j} \left\| \tilde{A}_j \tilde{B}_j^{-1} \varphi_j \right\|_{(1_j)} \right), \quad i = 3, 4. \tag{3.7}$$

Therefore, the scheme (1.1) is stable with respect to the operator coefficients.

(3) ⇒ (1). Now we suppose that the scheme (1.1) is stable with respect to the operator coefficients, i.e., the estimate (2.3) is valid, where the constants M_3 and M_4 satisfy inequalities (3.7).

We choose the perturbed operators $\tilde{B}_k$ and $\tilde{A}_k$ so as to ensure that $\left(B_k - \tilde{B}_k\right) u_k = 0$ and $\left(A_k - \tilde{A}_k\right) u_k = \delta_{k,k_0} \tau_{k_0+1}^{-1} B_k u_k$ for all $u_k \in H_h$ and $k = 0, \ldots, n_0 - 1$. Then $\delta_k^{(3)} = T_{n,k_0+1} \tilde{y}_{k_0}^{(3)}$ and

$$\sum_{k=0}^{n-1} \tau_{k+1} \left\| A_k - \tilde{A}_k \right\|_{(3_k)} = \sup_{\|\tilde{A}_{k_0} u_{k_0}\|_{(1_{k_0})} \neq 0} \frac{\|u_{k_0}\|_{(1_{k_0})}}{\left\| \tilde{A}_{k_0} u_{k_0} \right\|_{(1_{k_0})}} = \sup_{\|v_{k_0}\|_{(1_{k_0})} \neq 0} \frac{\left\| \tilde{A}_{k_0}^{-1} v_{k_0} \right\|_{(1_{k_0})}}{\|v_{k_0}\|_{(1_{k_0})}} = \left\| \tilde{A}_{k_0}^{-1} \right\|.$$

From the last relations and the estimate (2.3), we obtain the estimate

$$\left\| T_{n,k_0+1} \tilde{y}_{k_0}^{(3)} \right\|_{(1_{k_0})} \le M_3 \left\| \tilde{A}_{k_0}^{-1} \right\|,$$

and consequently, the operators $T_{n,k}$ are bounded for all $k = 1, \ldots, n-1$, since the operator $\tilde{A}_{k_0}$ is continuously invertible and hence $\|\tilde{A}_{k_0}^{-1}\| \le m$, $m = \text{const} > 0$. Condition (3.1) implies that the operators $T_{n,k}$ are also bounded for all $0 \le k \le n-1$ and $n = 1, \ldots, n_0$.

Since the solution of problem (2.4) is given by the formula $\delta_n^{(1)} = T_{n,k} \delta_k^{(1)}$ for each $k \le n-1$, we have the estimate (3.3) for $\delta_n^{(1)}$, which implies the uniform stability of the scheme (1.1) with respect to the initial data.

Note that, by using the method in [7], one can prove the equivalence of the notions of stability with respect to the initial data, stability with respect to the right-hand side, and coefficient stability without requiring that the operators $\tilde{A}_n$, $\tilde{A}_k$, and $\tilde{B}_k$ commute.

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