THE COMPARATIVE ANALYSIS OF VECTOR SPACE FILTRATIONS

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Let \((\Lambda, \leq)\) be a totally ordered set, \(\theta \subseteq \Lambda\), \(M = \{M_d\}_{d \in \theta}\), and \(X\) a vector space over the field \(\mathbb{K}\).

**Definition 1.** A map \(\lambda : X \to \Lambda\) is called a Lyapunov norm [1] or Lyapunov-Bogdanov [L.-B.] functional [2] if the following conditions hold: \(\lambda(cx) \leq \lambda(x), \lambda(x + y) \leq \max\{\lambda(x), \lambda(y)\}\) for every \(x, y \in X, c \in \mathbb{K}\).

By \(LB(X, \Lambda)\) denote the set of L.-B. functionals from \(X\) to \(\Lambda\). By \(R(\lambda)\) denote the range of functional \(\lambda\). Note that \(\lambda(0) = \min_{x \in X} \lambda(x)\) for every \(\lambda \in LB(X, \Lambda)\).

**Definition 2.** We say that \(M\) is a prefiltration of \(X\) if the following conditions hold:

\(i\) \(M_d\) is a subspace of \(X\) for every \(d \in \theta\);

\(ii\) \(M_{d_1} \subset M_{d_2}\) for every \(d_1, d_2 \in \theta\) such that \(d_1 < d_2\);

\(iii\) \(M_d \setminus \widetilde{M}_d \neq \emptyset\) for every \(d \in \theta\), where \(\widetilde{M}_d = \bigcup_{i < d} M_i\) if \(d \in \theta \setminus \min(\theta)\) and \(\widetilde{M}_d = \emptyset\) if \(d = \min(\theta)\).

**Definition 3** [2]. Prefiltration \(M\) of \(X\) is the \(\mathcal{B}\)-filtration of \(X\) if \(X = \bigcup_{d \in \theta} M_d\).

**Definition 4.** We say that the prefiltration \(M\) of \(X\) is the \(\mathcal{G}\)-filtration of \(X\) if the set \(\{d \in \theta \mid x \in M_d\}\) contains the least element for every \(x \in X\).

**Statement 1.** \(\mathcal{G}\)-filtration of \(X\) is a \(\mathcal{B}\)-filtration of \(X\).

**Statement 2.** \(\mathcal{B}\)-filtration \(M\) of \(X\) is \(\mathcal{G}\)-filtration of \(X\) if \((\theta, \leq)\) is a well ordered set.

Now we introduce the following notions associated with functional \(\lambda : X \to \Lambda\) [2]: \(\widetilde{X}_\lambda := \{x \in X \mid \lambda(x) \leq d\}\) is a Lebesque set of the functional \(\lambda\), \(\widetilde{X}_\lambda := \{\widetilde{X}_\lambda\}_{\lambda \in R(\Lambda)}\) is the condensed spectral family of the functional \(\lambda\).

**Theorem 1.** \(\widetilde{X}_\lambda\) is a \(\mathcal{G}\)-filtration of \(X\) for every \(\lambda \in LB(X, \Lambda)\).

**Theorem 2.** The map \(\lambda : X \to \theta\) such that \(\lambda(x) = d\) for every \(x \in M_d \setminus \widetilde{M}_d, d \in \theta\) is an L.-B. functional if \(M\) is \(\mathcal{G}\)-filtration of \(X\).

**Theorem 3.** Let \(\mathbb{I}\) be a set of positive irrational numbers, \(\theta := [0, +\infty) \cap \mathbb{Q}\), and \(X\) a vector space such that \(F := \{\lambda \in LB(X, \theta) \mid R(\lambda) = \theta\} \neq \emptyset\). Then the following assertions are valid for every \(\lambda \in F\):

\(i\) the family \(\{\widetilde{X}_\lambda\}_{\lambda \in R(\Lambda) \setminus \{q\}}\) is a \(\mathcal{B}\)-filtration of \(X\) for every \(q \in R(\lambda)\);

\(ii\) the family \(\{\widetilde{X}_\lambda\}_{\lambda \in R(\Lambda) \setminus \{q\}}\) is not \(\mathcal{G}\)-filtration of \(X\) for every \(q \in R(\lambda)\);

\(iii\) \(\bigcap_{i \in \mathbb{I}} \widetilde{X}_\lambda \neq \emptyset\) for every \(i \in \mathbb{I}\), where \(G(i) = \{q \in \theta \mid i \leq q\}\).

**Corollary 1.** There exists a \(\mathcal{B}\)-filtration \(M\) of some vector space \(X\) such that it is not a \(\mathcal{G}\)-filtration of \(X\) and its indexing set \((\theta, \leq)\) contains the least element.

In paper [2] it is shown that every \(\mathcal{B}\)-filtration of \(X\) is the condensed spectral family of some L.-B. functional on \(X\). But it is wrong. It follows from the theorem 1 and the corollary 1. However every \(\mathcal{G}\)-filtration of \(X\) is the condensed spectral family of some L.-B. functional on \(X\). It follows from the theorem 2.

The following corollary ensues directly from the theorems 1 and 3.

**Corollary 2.** There exists a \(\mathcal{G}\)-filtration \(M\) of some vector space \(X\) such that every its element is important for \(\mathcal{G}\)-filtration property of \(X\) by the family \(M\).