LIMIT DIFFUSIONS FOR MULTI-CHANNEL NETWORKS WITH INTERDEPENDENT INPUTS

E. Lebedev, A. Chechelnitsky
Kyiv National Taras Shevchenko University
Kyiv, Ukraine
leb@unicyb.kiev.ua

A process of information treatment in multi-channel stochastic networks with interdependent input flows are considered. In heavy traffic conditions functional limit theorems of diffusion approximation types are proved. Local characteristics of the diffusion are represented via network parameters.

Key words: multi-channel network, multi-dimensional Poisson input, diffusion approximation, uniform topology.

Methods of the theory of stochastic processes are an efficient tool for the analysis of the characteristics in the simulation of data transmission networks, computer networks and systems of collective use. This methods permit to evaluate the networks capacity, to find load reserves for network components and control parameters for information flows, as well as, e.g., to control the buffer memory of a node in a packet switching network. Adequate models of the real processes in networks of transmission and processing information are stochastic networks or networks of queues. Their structure is given by probability characteristics of input information flows, processing algorithms and packet-switching schemes.

The process of information treatment which is an object of our interest is a vector of large dimension with complex systems of stochastic relations which determine the process. In previous works, as a rule, input flows were independent. Thus, interdependence of service process components was caused by trajectory intersections for information packets before their outputs from the network. In the paper we omit this restriction and the process of information treatment became more complicated.

That is why the method of functional limit theorems is especially efficient for the analysis of information treatment in stochastic networks under consideration. The method gives the possibility to find those principles which form the foundation of a functioning process for a network of the given type, to construct an approximate process and to calculate the distribution of functionals in order to obtain integral characteristics for the functioning process.

By virtue of conditions imposed on model parameters in our case an approximate process will be diffusion and we will deal with development of the method of diffusion approximation for multi-channel stochastic networks. At first we consider a multi-channel queuing system with multi-dimensional Poisson input flow. The model consists of r nodes $E_1, \dots, E_r$. Each node operates as a queuing system $M/M/\propto$. It means that node E_i has a Poisson input flow of calls (exponential interarrival times) $y_i(t)$ with the parameter $\lambda_i > 0, i = 1, 2, \dots, r$. The calls arrive one at time and immediately take one of servers. Each node has an infinite number of servers. Correspondingly, the service time in the node $E_i, i = 1, 2, \dots, r$ has exponential distribution with the parameter $\mu_i > 0, i = 1, 2, \dots, r$. Each call can be served only in one node. After service completion calls leaves the nodes. In addition we suppose that the batch which consists of r calls arrive simultaneously at the all nodes according to a Poisson process $y(t)$ with the parameter $b > 0$. All nodes operate in parallel way in order to provide service in the batch of r calls. The construction of our model implies that in general the calls enter in our model according to a multi -dimensional Poisson input flow $(\nu_1(t), \dots, \nu_r(t))$ with parameter $\lambda_i > 0, \quad i = 1, 2, \dots, r, \quad b > 0$, and the component $\nu_i(t)$ has the following representation $\nu_i(t) = y_i(t) + y(t)$.

Such models may be used for planning modern computer networks. It allows to get a high speed and substantial economy of time.

Denote by $Q'(t) = (Q_1(t), \dots, Q_r(t)), t \in [0, T]$ the number of calls in the nodes of our model at time t. We will study the transient behavior of the multi-dimensional queuing process $Q(t)$ in heavy traffic conditions. This means that characteristics $\mu_i > 0, i = 1, 2, \dots, r$ of our model depend on a scaling parameter $n, n \to +\infty$ in the following way:

1) $\lim\limits_{n\to\infty} n\mu_i(n) = \mu_i \neq 0, \quad i = 1, 2, \dots, r.$

Denote by $Q^{(n)}(t) = (Q_1^{(n)}(nt), \dots, Q_r^{(n)}(nt)), t \in [0, T]$ the number of calls in the nodes of our model at time nt. We assume as well that the initial number of calls is 0 in each node:

2) $Q_i^{(n)}(0) = 0, \quad i = 1, 2, \dots, r.$

For process $\xi^{(n)}(t) = (\xi_1^{(n)}(t), \dots, \xi_r^{(n)}(t))$, where $\xi_i^{(n)}(t) = n^{-1/2}(Q_i^{(n)}(nt) - \alpha_i(t)n)$, $\alpha_i(t) = (\lambda_i + b)(1 - e^{-\mu_i t}), i = 1, 2, \dots, r$ (so $\xi^{(n)}(t)$ is a normalized queuing process $n^{-1/2}(Q^{(n)}(nt) - \alpha(t)n), \alpha(t) = (\alpha_1(t), \dots, \alpha_r(t)))$ we have proved a convergence in uniform topology to a diffusion process.

Theorem 1. *Suppose that conditions 1) and 2) hold. Then the sequence of processes $\xi^{(n)}(t) = (\xi_1^{(n)}(t), \dots, \xi_r^{(n)}(t))$ converges in uniform topology in any interval $[0, T]$ to the diffusion process $\xi(t)$, $\xi(0) = \xi_0 = 0$, with drift $a(x) = (a_1(x), \dots, a_r(x)) = (-\mu_1 x_1, \dots, -\mu_r x_r)$ and diffusion matrix*

$$\mathbf{B} = \begin{pmatrix} \lambda_1 + b)(2 - e^{-\mu_1 t}) & b & \dots & b \\ b & (\lambda_2 + b)(2 - e^{-\mu_2 t}) & b & b \\ \vdots & b & \ddots & b \\ b & \dots & b & (\lambda_r + b)(2 - e^{-\mu_r t}) \end{pmatrix}$$

Note that the investigation of heavy traffic convergences has a long history and there are several approaches oriented to different classes of queuing models. We prove our theorem using a local approach because it is more convenient for our model. This approach was developed in [1].

The Theorem 1 allows to study instead of functionals of the complex service process (for example, total income connected with service in the network) the corresponding functionals of the diffusion process.

Now we consider more complicated models of multi-channel networks which operate in the following mode.

The network consists of r nodes of information treatment. At the i-th node from the outside information packets arrive in the $\tau_k^{(i)}$, $\quad k = 1, 2, \ldots$, moments of time, $\nu_i(t)$ is the total number of packets arrived in the time interval $[0, t]$. Each of r nodes is a multi-channel system to treat packets of information. If a packet arrives in such system then its treatment begins immediately. For the node i the service time is exponentially distributed with parameter $\mu_i, i = 1, 2, \ldots, r$. The direction of packet motion in the network interior is given by a switching matrix $P = \|p_{ij}\|_1^r$. For any $i = 1, 2, \ldots, r$ $p_{ir+1} = 1 - \sum_{j=1}^r p_{ij}$ is the exit probability for the packet treated in the i-th node. In the used system of notation this network is coded by the symbol $[G|M|\propto]^r$.

As before we will define the process of information treatment in the $[G|M|\propto]^r$-network as an r - dimensional process $Q^{'}(t) = (Q_1(t), \ldots, Q_r(t)), t \geq 0$, where $Q_i(t)$ is the number of information packets in the i-th node at the instant of time t .

Let us study the process of information treatment $Q(t)$ in heavy traffic regime. Heavy traffic regime means: parameters of the network depend on n (the number of series) in such a way that the conditions 1), 2) hold true and the input flow is closed to Brownian motion.

3) There are constants $\lambda_i \geq 0, i = 1, 2, \ldots, r, \quad \lambda_1 + \ldots + \lambda_r \neq 0$, that

$$n^{-1/2}(\nu_1^{(n)}(nt) - \lambda_1 nt, \ldots, \nu_r^{(n)}(nt) - \lambda_r nt) \Rightarrow_{n\to\infty}^U W^{'}(t) = (W_1(t), \ldots, W_r(t)),$$

where $W(t)$ is a r-dimensional process of Brownian motion with the null - vector of mean values $EW(1) = 0$ and the correlation matrix $EW(1)W'(1) = \sigma^2 = \|\sigma_{ij}^2\|_1^r$, symbol $\Rightarrow^U$ means weak convergence in uniform topology.

The other parameters of the $[G|M|\infty]^r$ - network do not depend on n.

In the context of conditions 1) - 3) for the open $[G|M|\infty]^r$ - network we shall consider the sequence of stochastic processes

$$\xi^{(n)}(t) = n^{-1/2}(Q^{(n)}(nt) - nq(t)), \quad t \geq 0,$$

where $q^{'}(t) = (q_1(t), \ldots, q_r(t)) = (\theta/\mu)'(I - P(t))$, $(\theta/\mu)' = (\theta_1/\mu_1, \ldots, \theta_r/\mu_r)$, $\theta' = (\theta_1, \ldots, \theta_r) = \lambda'(I - P)^{-1}$ - is a solution of the balance equation for the $[G|M|\infty]^r$ - network, $\lambda' = (\lambda_1, \ldots, \lambda_r)$, $P(t) = \|p_{ij}(t)\|_1^r = exp(Qt)$, $Q = \Delta(\mu)(P - I), \Delta(x) = \|x_i\delta_{ij}\|_1^r$ is a diagonal matrix with the vector on the principal diagonal.

The process $\xi^{(n)}(t)$ may be approximated by diffusion in the following way.

Theorem 2. *Let for a stochastic network of the type $[G|M|\infty]^r$ conditions 1) - 3) be satisfied and the spectral radius of the switching matrix P be strictly less than 1 . Then for any finite interval $[0, T]$ the sequence of stochastic processes $\xi^{(n)}(t), \quad n \geq 1$, converges weakly to the diffusion $\xi(t) \quad (\xi(0) = 0)$ with drift $A(x) = Q^{'}x$ and diffusion matrix $B(t) = \Delta[q^{'}(t)Q] - Q^{'}\Delta[q(t)] - \Delta[q(t)]Q + \sigma^2$ in the uniform topology, as $n \to \infty$.*

A proof of the theorem use a representation of information treatment process as a sum of the indicators on a path of the input flow.

The Theorem 2 extends the results on diffusion approximation of multi-channel networks from [2] by means of restriction removal on input flows.

In summary we make a comment. In our case the switching matrix does not depend on the series parameter n. But if we extend the research field and put

$$P = P_n = P_0 + n^{-1}B_0 + o(n^{-1}),$$

where $P_0 = \|\delta_{\alpha\beta}P^{(\alpha)}\|_1^{r_0}$, $P^{(\alpha)} = \|p_{ij}^{(\alpha)}\|_{i,j\in I_\alpha}$ is an indecomposable stochastic matrix then in the process of approximation it is possible to enlarge the nodes of the initial network: $r \to r_0$, $r_0 \leq r$. The nodes of the set I_α are then joined in a node "α" .

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