Let f be an invertible function. A parastrophe ${}^{\sigma}f$ of f is defined by

$$ {}^{\sigma}f(x_{1\sigma}; x_{2\sigma}) = x_{3\sigma} \iff f(x_1; x_2) = x_3 $$

for any $\sigma \in S_3$, where $S_3 := \{\varepsilon, \ell, r, s, \ell s, rs\}$ and $s := (12)$, $\ell := (13)$, $r := (23)$.

Theorem. *Let f_1, f_2, f_3, f_4 be binary invertible functions defined on a set Q. Then a quadruple $(f_1; \ldots; f_4)$ is a solution of*

$$ F_1(y; y) = F_2(x; F_3(x; F_4(x; x))) $$

iff there exists a substitution α and an element a of Q such that

$$ f_1(y; y) = a, \quad f_2(x; \alpha x) = a, \quad f_4(x; x) = {}^{r}f_3(x; \alpha x). $$

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SYSTEMS OF LINEAR CONGRUENCES, BALANCED MODULAR LABELLINGS OF GRAPHS AND CHROMATIC TOTIENTS

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The present research is devoted to some number-theoretic consequences of certain notions and results of algebraic graph theory. Given a finite simple connected graph $G = (V, E)$, an orientation of its edges and a natural number k, we consider edge k-labellings $f : E \to \mathbb{Z}_k^*$ satisfying Kirchhoff's circuit law, where $\mathbb{Z}_k^*$ is the set of invertible elements of the ring $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. In terms of variables $x_e = f(e)$, $e \in E$, we consider the system of homogeneous linear congruences modulo k which correspond to the (simple, independent) cycles of G, have coefficients 0 and ± 1 (moreover, their matrix is unimodular) and are subject to the 'side condition' that all values of their variables are coprime with k. The solutions with the latter property are called *invertible.* The choices of edge orientations and independent cycles do not influent the resulting system of congruences up to equivalence. Let $R(G, k)$ be the number of invertible solutions of such a system.

Theorem. *For any finite connected graph G, $R(G, k)$ is the multiplicative arithmetic function of k that is determined by the formula*

$$ R(G, p^a) = \chi(G, p) p^{(a-1)(n-1)-1} \tag{1} $$

for every prime p and integer $a \geqslant 1$, where $\chi(G, z)$ is the chromatic polynomial of G and $n = |V|$ is the number of vertices.

This basic equation shows that $R(G, k)$ is a kind of totient functions [1], which we call a *chromatic totient.* In particular, $R(K_2, k) = \phi(k)$, Euler's totient function, where $K_2 =$ •—•

is the graph consisting of two vertices and one edge. When G is a cycle, the general formula turns into the well-known formula of Rademacher — Brauer [2, Ch. 3] for the number of invertible solutions of the congruence $x_1 + \cdots + x_n \equiv 0 \pmod{k}$ (cf. also the concluding remark in [3]).

Corollary. *The system of congruences corresponding to the cycles of G has an invertible solution modulo k if and only if $p \geqslant \lambda(G)$ for all prime p dividing k, where $\lambda(G)$ is the chromatic number of G.*

We refer to [4] for useful details concerning the chromatic polynomials and numbers of graphs.

There are three main ingredients of the proof:

- the existence of a bijection between proper vertex p-colorings of a rooted connected graph and nowhere-zero $\mathbb{Z}_p$-labellings of its edges that satisfy Kirchhoff's second law;
- the equivalence of the restrictions $\gcd(p, i) = 1$ and $i \not\equiv 0 \pmod{p}$ for any prime p;
- the familiar fact (see, e.g., [5]) that in a *fundamental cycle base* $\mathcal{B}$ of G, every cycle C contains an exclusive edge (such is, e.g., the edge $e \in G - T$ by which C is determined as the unique cycle of the subgraph $e \cup T$, where $T = T_{\mathcal{B}}$ is the corresponding spanning tree of G).

Formula (1) extends easily to disconnected graphs. A generalization to non-homogeneous systems of linear congruences where some variables get prescribed values holds with an appropriate "partial" chromatic polynomial instead of $\chi(G, z)$.

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GROUPS WITH PRESCRIBED PROPERTIES OF FINITE SUBGROUPS GENERATED BY COUPLES OF 2-ELEMENTS

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We discuss results of the research started in [1] and continued in [2–4].

Theorem. *Suppose that in a group G the order of the product of every two involutions is finite. If every finite subgroup of G generated by a couple of 2-elements is either nilpotent of class at most 2 or has an exponent dividing 4, then all 2-elements of G form a normal subgroup which is either nilpotent of class at most 2 or has an exponent dividing 4.*

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