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Constructive methods for factorization of matrix-functions

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A survey of constructive methods for the factorization of $n \times n$ matrix functions is presented. The importance of these methods for theoretical and practical applications is singled out. Several classes of matrices are considered which are factorized by the proper technique. The perspective of the constructive methods and procedures is discussed and open questions are formulated.

Keywords: matrix-functions, factorization, constructive methods

AMS 2010 Classification: Primary: 15A23; Secondary: 15A54, 15B05, 30E25, 34M50, 45E10

1. Introduction

This paper deals with the problem of factorization of matrix-functions. Let us formulate this problem in the classical setting.

Let Γ be an oriented simple closed smooth curve on the complex plane $\mathbb{C}$ (the special case $\Gamma = \mathbb{R}$ can be considered too). Denote by D^+, D^- the domains on the Riemann sphere lying, respectively, to the left and to the right of the curve Γ, with respect to the chosen orientation. Let $G \in \mathscr{G}(\mathscr{M}(\Gamma)^{n\times n})$[1]

$$G : \Gamma \to \mathscr{M}^{n\times n}$$

be a non-singular matrix function, defined on Γ (for example, a matrix with continuous entries). The matrix-function $G = G(t)$ admits a *(right) factorization* if it can be represented in the form

$$G(t) = G^-(t)\Lambda(t)G^+(t), \tag{1..1}$$

where non-singular matrices $G^-(t), G^+(t)$ possess an analytic continuation into D^-, D^+, respectively,[2] $\Lambda(t)$ is the $n \times n$ diagonal matrix,

$$\Lambda(t) = diag\left\{ \left(\frac{t-t^+}{t-t^-}\right)^{æ_1}, \ldots, \left(\frac{t-t^+}{t-t^-}\right)^{æ_n} \right\}, \tag{1..2}$$

and $t^+ \in D^+, t^- \in D^-$ are certain (fixed) points. In particular, if $\Gamma = \mathbb{R}$, then one can choose

$$t^+ = i, t^- = -i,$$

[1]Here $\mathscr{M}(\Gamma)^{n\times n}$ stands for a set of square $n \times n$ matrix functions, $\mathscr{G}$ means invertibility.

[2]Since $(G^-)^{\pm 1}$, $(G^+)^{\pm 1}$ possess analytic continuation into corresponding domains, then both matrices $G^-(z)$, $G^+(z)$ need to be non-singular in D^-, D^+, respectively.

and if Γ is a bounded curve and $0 \in D^+$, then

$$\frac{t - t^+}{t - t^-} = t.$$

The integer numbers $æ_1, \ldots, æ_n$ are called *partial indices*, and the matrices $G^-(t)$, $G^+(t)$ are known as *minus-, plus-factors*. If factorization (1..1) exists, then the partial indices are uniquely determined up to their order, i.e. they are *invariants of the factorization problem*. Upon interchanging $G^-(t)$ and $G^+(t)$ in (1..1) we arrive at the *left-factorization*.

Initially, the factorization problem is linked to Riemann or, more precisely, with two problems formulated by him, known as the *Riemann boundary value problem* (or *Riemann-Hilbert boundary value problem*, see Gakhov (1977)), and the *Riemann monodromy problem* (or *21st Hilbert problem*, or *Riemann-Hilbert problem*, see Bolibrukh (1990)). In fact, the notion of partial indices was first introduced in the study of the vector-matrix Riemann boundary value problem in Muskhelishvili & Vekua (1943).

In the present day, the factorization problem is interesting due to its connections to notable mathematical problems (vector-matrix boundary value problems, systems of singular integral equations, the Wiener-Hopf and other convolution type equations, the Riemann-Hilbert problem, classification of vector bundles on the Riemann sphere, nonlinear evolution equations, the Toeplitz operators, etc), as well as to applied problems (elasticity and elasto-plasticity, radiation and neutron transport, wave diffraction, fracture mechanics, geomechanics, signal processing, financial mathematics, etc). Several monographs and extended surveys on the theory, on specific approaches and applications of the factorization of matrix-functions has been published (see, e.g., Bart *et al.* (2008), Böttcher & Spitkovsky (2013), Clancey & Gohberg (1987), Ehrhardt & Speck (2002), Ehrhardt & Spitkovsky (2001), Gohberg *et al.* (2003), Lawrie & Abrahams (2007), Litvinchuk & Spitkovsky (1987), Vekua (1967)).

Sometimes the factorization problem is called the Wiener-Hopf factorization. The latter is connected with the so-called Wiener-Hopf technique developed initially for the study of the Wiener-Hopf integral equation of the form

$$\int_0^{+\infty} k(x-t) f(t) dt = g(x), \quad 0 < x < +\infty.$$

The fundamentals of the idea were initially proposed in the original paper by Wiener & Hopf (1931) and is outlined as follows: by applying a Fourier transform to the integral equation, we can derive the so-called Wiener-Hopf functional equation

$$K(\alpha)F^+(\alpha) + F^-(\alpha) = G(\alpha),$$

where unknown functions F^+, F^- have to be analytic in the half-planes D^+, D^- intersecting along the strip $\mathscr{D}$, and the given functions $K(\alpha)$, $K^{-1}(\alpha)$, $G(\alpha)$ are analytic in this strip (called *the Wiener-Hopf strip*). For many mixed boundary value problems the Wiener-Hopf formulation appears in the form of coupled Wiener-Hopf equations that can be reduced to a functional equation in the matrix form (see Noble (1988)) and thus is related to the above described factorization problem (see also Duduchava (1979), Duduchava & Wendland (1995) in relation to crack problems). The historical development of the Wiener-Hopf method is presented in Lawrie & Abrahams (2007). The relationship between the Wiener-Hopf factorization and the Riemann boundary value problem has been recently discussed in Kisil (2015).

Generalizations of the classical factorization problem have also been proposed (see, e.g., Simonenko (1968), Veitch & Abrahams (2007)). Among these generalizations we point out the so-called *Φ-factorization*, based on the ideas of Simonenko (1968) and dealing initially with the factorization of matrices with Lebesgue-measurable entries.

The Φ factorization was introduced by I. Spitkovsky (see Litvinchuk & Spitkovsky (1987)). Below we repeat this definition following the survey paper Ehrhardt & Spitkovsky (2001), where this definition is given in a complete form suitable for our presentation.

Let G be $n\times n$ matrix function, $G\in (L^\infty(\Gamma))^{n\times n}$ and the following assertions hold:

(i) $G^+ \in \left(L^q_+(\Gamma)\right)^{n\times n}$, $(G^+)^{-1} \in \left(L^p_+(\Gamma)\right)^{n\times n}$;

(ii) $G^- \in \left(L^p_-(\Gamma)\right)^{n\times n}$, $(G^-)^{-1} \in \left(L^q_-(\Gamma)\right)^{n\times n}$;[3]

The operator

$$f \to (G^+)^{-1}\Lambda^{-1}P_\Gamma\left((G^-)^{-1}f\right) \tag{1..3}$$

is a well defined linear mapping from the linear space of all rational vector functions with poles off Γ and maps this set into $\left(L^1(\Gamma)\right)^n$. Let, in addition to conditions (i), (ii), we assume that

(iii) mapping (1..3) is bounded in the L^p-norm (and thus it can be extended by continuity to a bounded operator on the whole space $(L^p(\Gamma))^n$).

In this case, the representation (1..1) is called a Φ-factorization in the space $L^p(\Gamma)$.

Here $p^{-1}+q^{-1}=1$ and the sub-indices "$+$" or "$-$" in notation referring to Lebesgue spaces signify that the corresponding functions (vector-functions or matrix-functions) are analytically continued into "$+$"- or "$-$"-domains with boundary functions (vector-functions or matrix-functions) in the corresponding Lebesgue spaces, the operator

$$P_\Gamma = 1/2(I+S_\Gamma)$$

is a projector,[4] and the operator

$$(S_\Gamma f)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)}{\tau - t}d\tau, \quad t\in\Gamma.$$

is the singular integral operator on Γ.

If the matrix G admits the Φ-factorization, then the partial indices $æ_1,\dots,æ_n$ are also defined uniquely up to their order. The results on Φ-factorization (as well as on their generalizations) are discussed in several monographs (see, e.g., Bart *et al.* (2008), Clancey & Gohberg (1987), Litvinchuk & Spitkovsky (1987), Speck (1985)) and survey papers (see, e.g., Böttcher & Spitkovsky (2013), Castro *et al.* (2005), Gohberg *et al.* (2003)). In particular, in Castro *et al.* (2005), it has been proposed a criterion for some classes of continuous matrix functions on the real line with a jump at infinity to be factorized as in the classical sense as in the sense of the asymmetric factorization. Under asymmetric (antisymmetric) factorization it is understood the representation $G=G^-\Lambda G^e$ $(G=G^-\Lambda G^o)$, where G^e is even on the real line (G^o is odd on the real line). The result of Castro *et al.* (2005) yields the existence of generalized inverses of matrix Wiener-Hopf plus Hankel operators and provides precise information about the asymptotic behavior of the factors at infinity and of the solutions to the corresponding equation at the origin.

[3]Here $\left(L^{p,q}_+(\Gamma)\right)^{n\times n}$, $\left(L^{p,q}_-(\Gamma)\right)^{n\times n}$ mean sets of $n\times n$ matrices with entries in the corresponding Lebesgue spaces which possess analytic continuation into corresponding domains. It follows, in particular, that $G^-(z)$, $G^+(z)$ need to be non-singular in D^-, D^+, respectively.

[4]Which is, in fact, not necessarily orthogonal.

The factorization problem has various formulations, numerous aspects, and is linked to many problems of Analysis. We will now point out some books dedicated to the different points of view in studies of the factorization problem.

To begin, the classical book by Vekua (1957) presented the method of the systems of the singular integral equations to study the factorization. The most extended variant of this theory is presented in the second Russian edition (1970), see also Muskhelishvili (1968) (English translation of Vekua's book is based on the first Russian edition of 1950).

The factorization problem for continuous or piecewise continuous matrix functions (and, in general, for measurable matrix functions) was thoroughly studied and was presented, for example in the book by Litvinchuk & Spitkovsky (1987). This presentation touches both the classical and generalized factorizations, and deals mainly with theoretical development in the subject area.

The book by Clancey & Gohberg (1987) presents a systematic study of the factorization problem from the point of view of the theory of singular integral operators (see Gohberg & Krupnik (1991-1992)). In this book the emphasis is on the connections between factorization relative to a contour and singular integral operators. To see these connections in more detail the authors consider on $L_p^n(\Gamma)$ the singular integral operator of the form $T = BI + CS_\Gamma$, where B, C are given $n \times n$ matrices defined on Γ, and I, S_Γ are, respectively, unity and singular integral operators on Γ. The main focus of the book is on generalized factorization, but the classical version (referred to as "continuous factorization") also features in the discussion.

Based on the ideas of Clancey & Gohberg (1987), the book Bart *et al.* (2008) considers various types of the factorization problems for matrix and operator functions. The problems appear in different areas of mathematics, its applications, and a unified approach to treat them is developed. The main theorems yield the explicit necessary and sufficient conditions for the factorizations to exist and explicit formulas for the corresponding factors, stability of the factors relative to a small perturbation of the original function is also studied. The unifying theory developed in the book is based on a geometric approach which has its origins in different fields. A number of initial steps can be found in: (1) the theory of non self-adjoint operators, where the study of invariant subspaces of an operator is related to factorization of the characteristic matrix or operator function of the operator involved, (2) mathematical systems theory and electrical network theory, where a cascade decomposition of an input-output system or a network is related to a factorization of the associated transfer function, and (3) the factorization theory of matrix polynomials in terms of invariant subspaces of a corresponding linearization. In all three cases a state space representation of the function to be factorized is used, and the factors are also expressed in terms of state space form.

We also mention a few specialized books, namely Speck (1985), which considers the Wiener-Hopf method and Wiener-Hopf factorization (see also Noble (1988)), and Böttcher *et al.* (2002) which examines the factorization of almost periodic matrix functions.

It seems almost impossible to discuss all the questions related to the factorization problem. Therefore, in this survey paper we restrict our attention to the constructive methods of factorization in its classical formulation. This area is less developed with respect to theoretical development of the subject, but it is one of the great practical importance. We present most of the known results in this area, and acknowledge that the description could be extended in many directions (see Sec. 11.).

The paper is organized as follows. In Sec. 2. we present a number of general results known in the factorization theory. Later in the paper we classify the constructive results either by the developed methods or by the specific classes of matrix functions to which these methods are applied. The corresponding Sections constitute the main part of the paper. We conclude in Sec. 11. with the discussion of the perspectives of the constructive approaches and of further developments in the area of the matrix functions factorization. We present a large but non-exhaustive list of references, and note that many other sources can be found via cross-references.

2. Few general results

Some general results exist for classical-type factorization. We formulate them here for readers' convenience.

- 1. The necessary condition for a continuous matrix-function $G:\Gamma\to\mathbb{C}^{n\times n}$ to be factorized is its non-singularity:
$$\det G(t)\neq 0,\quad \forall t\in\Gamma.$$
- 2. If the matrix admits a factorization, then the partial indices are defined uniquely up to the order (see, e.g., Vekua (1967)). Thus one can always suppose $æ_1\geqslant\cdots\geqslant æ_n$.
- 3. If a continuous matrix G admits a factorization, then the sum of partial indices is equal to the index (winding number) of the determinant of the matrix (see, e.g., Gakhov (1950)):
$$æ_1+\cdots+æ_n = ind_\Gamma \det G(t) = wind_\Gamma \det G(t).$$
- 4. The partial indices are stable iff $æ_1-æ_n\leqslant 1$. In particular, if one approximates a given matrix G by factorizable sequence of matrices G_k, then in the limit, as $k\to\infty$, the factors G_k^+, G_k^- (as well as corresponding partial indices $æ_j^{(k)}$) do not necessarily give a factorization of the initial matrix (and its partial indices $æ_j$). This criterion for the stability was proved for special classes of matrices in Shmulian (1953), Shmulian (1954), Chebotarev (1956), and in general form in Bojarsky (1958), Gohberg & Krein (1958b), see also Litvinchuk (1967), Spitkovskij (1974), Tishin (1988).
- 5. If it exists, the factorization of a matrix-function (i.e. the factors G^+, G^-) is defined non-uniquely. The factors are determined up to certain rational block-triangular matrix-functions (see, e.g., Ehrhardt & Spitkovsky (2001)).
- 6. The factors of a canonical factorization are determined uniquely up to a constant multiple (see, e.g., Gohberg & Krein (1958a)).
- 7. If $n=1$, then any piece-wise Hölder-continuous matrix (function) can be factorized explicitly (see, e.g., Gakhov (1977)).
- 8. Any non-singular Hölder-continuous matrix function admits a continuous factorization (see, e.g., Clancey & Gohberg (1987), Vekua (1967)).
- 9. If any 2×2 matrix function that is positive and Hölder continuous on Γ admits the (continuous) canonical factorization, then Γ is a circle (see Markus & Matsaev (1994)).
- 10. The continuous factorization of a continuous matrix-function $G\in(\mathscr{C}(\Gamma))^{n\times n}$ may not exist since the algebra of continuous functions is not decomposable (see, e.g., Clancey & Gohberg (1987)).

- 11. Let $n \geqslant 2$. Given any two vectors $\rho, \lambda \in \mathbb{Z}^n$, such that $\sum \rho_j = \sum \lambda_j$, there exists a matrix $A(t) \in \mathscr{G}(\mathscr{R}(\Gamma))^{n\times n}$ (i.e. invertible on Γ, possessing rational extension to the whole complex plane), whose vectors of partial indices for the right- and left-factorization (right and left partial indices) are equal to ρ, λ, respectively (see Feldman & Markus (1998)).

- 12. A nonsingular continuous matrix-function always admits a factorization in $L^p(\Gamma), 1 < p < \infty$, and this factorization can be made independent of $p, 1 < p < \infty$ (see, e.g., Clancey & Gohberg (1987)).

3. Triangular matrix functions

Triangular matrix functions are supposed to form the most simplest class of matrices for which it is possible to obtain explicit factorizations (see, e.g. (Clancey & Gohberg, 1987, Ch. 4)). Two main methods are described in Clancey & Gohberg (1987). One of them is a constructive procedure, which is due to Gohberg & Krein (1958a), to obtain a continuous factorization of a non-singular triangular matrix function with the continuous entries from a decomposing algebra. Another approach is the general rule of Chebotarev (1956) to get the partial indices of a 2×2 triangular matrix function (which is described above in Sec. 4.). In fact, apart from the factorization procedure applied to triangular matrices, the interesting question is how to single out a class of matrices which can be reduced to the triangular form by using rational transformation. As a matter of fact, the class of matrix functions G which can be transformed by a (more general) transformation of the form

$$G(t) \mapsto \hat{G}(t) = U(t)G(t)V(t), \quad U \in \mathscr{G}\mathscr{R}^{2\times 2}, \quad V \in \mathscr{G}\mathscr{R}^{2\times 2}, \tag{3..1}$$

into a triangular matrix coincides with the class described in Spitkovskij & Tashbaev (1989).

More generally, it is important with respect to all factorization problems to describe transformations of the form

$$G(t) \mapsto \hat{G}(t) = U_-(t)G(t)V_+(t), \quad U_- \in \mathscr{G}\mathscr{R}_-^{2\times 2}, \quad V_+ \in \mathscr{G}\mathscr{R}_+^{2\times 2}, \tag{3..2}$$

provided that $\hat{G}$ belongs to a class of matrix functions for which a factorization theory is known (see Ehrhardt & Speck (2002)). Here $\mathscr{R}_\pm^{2\times 2}$ are classes of rational 2×2 matrix functions with no pole in $D^\pm$.

The idea of systematically considering transformations (3..2) in factorization theory was first taken up by Spitkovskij & Tashbaev (1989). They gave a description of all 2×2 matrix functions which can be transformed by certain transformation (3..2) into a triangular matrix. Such matrix functions have at most three rationally independent entries, and the rational dependence between the entries is of a specific nature.

4. Chebotarev-Gakhov method

The factorization problem is tightly connected to the (vector-matrix) Riemann boundary value problem (see, e.g., Gakhov (1977), Muskhelishvili (1968))[5] for several unknown functions. This consists of the determination of two vector-functions $\Phi^+(z), \Phi^-(z)$, analytic in domains D^+, D^-, respectively, whose boundary value satisfy the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{4..1}$$

[5] This problem is also known as the Hilbert boundary value problem or $\mathbb{C}$-linear conjugation problem.

where $G(t)$ and $g(t)$ are given matrix- and vector-function on the contour Γ[6]. The homogeneous problem (i.e. when $g(t) \equiv 0$) is very similar to the factorization problem (1..1) even in its form. A special case of problem (4..1) was posed by Riemann (1876) in his work on (complex) differential equations with algebraic coefficients in connection with the construction of a differential equation whose solutions admit a given linear substitution when its variable is encircling a given family of ("singular") points (i.e. an equation with a given monodromy group). This question on possibility to construct differential equations with a given monodromy group is known as the Riemann-Hilbert problem or 21-st Hilbert problem (for a more exact formulation of this problem and an extended description of corresponding results see Bolibrukh (2009), Ehrhardt & Spitkovsky (2001)).

It is well-known that in the classical setting (for Hölder continuous G, g and $G \neq 0$) the scalar Riemann boundary value problem (i.e. for $n = 1$) possesses an explicit solution. The so-called *canonical function* $X^{\pm}(z)$ takes a central role in this solution,

$$X^{+}(z) = \exp\gamma^{+}(z), \quad X^{-}(z) = z^{-æ}\exp\gamma^{-}(z),$$

$$\gamma^{\mp} = \mp\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\log[\tau^{-æ}G(\tau)]}{\tau - z}d\tau,$$

where

$$æ = ind_{\Gamma}G(t) = wind_{\Gamma}G(t) = \frac{1}{2\pi}\Delta_{\Gamma}\arg G(t)$$

is the index of the coefficient $G(t)$ (or its winding number). For nonnegative index the homogeneous problem has $æ + 1$ linear independent solutions

$$\Phi^{\pm}(z) = z^{k}X^{\pm}(z),\ k = 0, 1, \dots, æ,$$

but for negative index the homogeneous problem has no analytic solution. Sometimes the solution formulas (both for homogeneous and inhomogeneous problems) are called *Gakhov's formulas*.

Using the canonical function and Sokhotsky-Plemelj formulas (see, e.g., Gakhov (1977), Muskhelishvili (1968)), one can immediately construct a solution of the factorization problem

$$G^{\mp}(t) = \exp\left\{\frac{\log[t^{-æ}G(t)]}{2} \mp \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\log[\tau^{-æ}G(\tau)]}{\tau - t}d\tau\right\},\ \Lambda(t) = t^{æ}. \tag{4..2}$$

Gakhov (see Gakhov (1952)) posed the question of how to describe an as large as possible class of matrix-functions for which the homogeneous Riemann boundary problem can be solved by using formulas similar to (4..2)

$$\Phi^{\pm}(z) = \exp\left\{\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\log G(\tau)}{\tau - z}d\tau\right\}. \tag{4..3}$$

To begin, we introduce a suitable definition for a function of a matrix. Since we have no advance knowledge of the behaviour of solution at ∞, this solution is "weaker" than that of the case n=1. Finally, it is necessary that the involved matrices (say $A(t), B(t)$) satisfy the relation

$$\exp\{A(t)\}\cdot\exp\{B(t)\} = \exp\{A(t) + B(t)\},$$

[6] Here we consider only the case when the contour Γ is a simple bounded closed curve.

which is clearly not always the case for matrix-functions.

Such a class of functions was found and completely characterized in Chebotarev (1956), Chebotarev (1956a). These are functionally-commutative matrix-functions, i.e. those matrix functions which satisfy the following relation

$$G(t)G(\tau) = G(\tau)G(t), \ \forall t, \tau \in \Gamma. \tag{4..4}$$

In particular, it was shown that by linear transformation (with constant matrices) a 2×2 functionally-commutative matrix can be reduced to a triangular form. Moreover, by such reduction the functionally-commutative matrix can be transformed to one of the simplest form (containing the lowest number of arbitrary independent entries). In particular, for lower order matrices, these simplest forms are the following (for general description we refer to Chebotarev (1956a)):

- in the case $n = 2$:

$$G(t) = \begin{pmatrix} \varphi_1(t) & 0 \\ 0 & \varphi_2(t) \end{pmatrix}, \ G(t) = \begin{pmatrix} \varphi_1(t) & 0 \\ \varphi_2(t) & \varphi_1(t) \end{pmatrix};$$

- in the case $n = 3$:

$$G(t) = \begin{pmatrix} \varphi_1(t) & 0 & 0 \\ 0 & \varphi_2(t) & 0 \\ 0 & 0 & \varphi_3(t) \end{pmatrix}, G(t) = \begin{pmatrix} \varphi_1(t) & 0 & 0 \\ \varphi_2(t) & \varphi_1(t) & 0 \\ 0 & 0 & \varphi_3(t) \end{pmatrix};$$

$$G(t) = \begin{pmatrix} \varphi_1(t) & 0 & 0 \\ \varphi_2(t) & \varphi_1(t) & \beta\varphi_3(t) \\ \alpha\varphi_3(t) & 0 & \varphi_1(t) \end{pmatrix}, G(t) = \begin{pmatrix} \varphi_1(t) & 0 & 0 \\ \varphi_2(t) & \varphi_1(t) & 0 \\ \varphi_3(t) & \varphi_2(t) & \varphi_1(t) \end{pmatrix}.$$

For the piece-wise continuous functionally-commutative matrices, a finite algorithm for the solution of the homogeneous Riemann boundary value problem was proposed consisting of the following steps:
1) transformation to a block-triangular form;
2) determination of the proper branches of the solution (4..3);
3) refinement of the above solution by polynomial transformation to a quasi-bounded (i.e. possibly having logarithmic singularities) analytic solution.

In particular, if the coefficient $G(t)$ of the problem is a continuous functionally-commutative matrix, then by this algorithm we arrive at the so-called *canonical family of solutions* to the homogeneous Riemann boundary value problem. This means that if we take all the solutions as columns, then this matrix is non-singular at any finite point and the maximal order at infinity of elements of a column is equal to one of the partial indices of the coefficient matrix, i.e. $k_l = -æ_j$.

In Chebotarev (1956) this algorithm was realized for all possible situations in the case of 2×2 matrices and the partial indices were explicitly calculated (this approach was recently generalized for matrices of the higher order by Primachuk (2015)).

In principle, Chebotarev's approach is is consistent with the general theory of of solving the vector-matrix boundary value problem (see Muskhelishvili (1968), Vekua (1967)). Specifically, if we know the *normal family of solutions* to the problem (i.e. the constituent non-singular matrix at each finite point), then it is possible to construct a canonical family by polynomial transformation. In the case considered by Chebotarev (functionally-commutative matrices), the normal system of solutions is constructed by Gakhov's formula, and the partial indices (for 2×2 matrices) calculated explicitly.

The Chebotarev-Gakhov method was successfully developed in Kiyasov (2008), Kiyasov (2012) (as well as in Primachuk (2015)). In Kiyasov (2013) this method was combined with the method used in theory of nonlinear boundary value problems.

5. Analytic matrix functions

In Adukov (1999), a new method for calculation of the partial indices and the Wiener-Hopf factorization of analytic matrix-valued functions was proposed. It has computational advantages with respect to an earlier method proposed by the author. This method is used to find the divisors of an analytic matrix-valued function $A(t)$ that generate the zeros of $\det A(t)$.

The following class of the matrix functions is considered, where $A(t)$ is a matrix-valued function that is continuous and invertible on the curve Γ and is analytic in the domain D^+. Denoting $\Delta(t) = \det A(t)$ it is supposed that the Wiener-Hopf factorization of this determinant exists

$$\Delta(t) = \Delta_-(t)t^{æ}\Delta_+(t).$$

The following assertions follow from Adukov (1993):

1. the partial indices of left-factorization (left partial indices) $\lambda_1,\ldots,\lambda_p$ and the right partial indices $\rho_1,\ldots,\rho_p$ of $A(t)$ are not negative;

2. the row $r_j(t) := \Delta_-(t)[R_-^{-1}(t)]_j$ is a vector-valued polynomial in t^{-1} whose degree is not greater than $æ - \rho_j$; here $R_-(t)$ is the factor of the right-factorization $A(t) = R_-(t)\Lambda_r(t)R_+(t)$;

3. the column $l_j(t) := \Delta_-(t)[L_-^{-1}(t)]^j$ is a vector-valued polynomial in t^{-1} whose degree is not greater than $æ - \lambda_j$; here $L_-(t)$ is the factor of the left-factorization $A(t) = L_+(t)\Lambda_l(t)L_-(t)$.

In Adukov (1999), the partial indices of the matrix function $A(t)$ are calculated in terms of moments of the matrix-valued function $\Delta_-^{-1}(t)A(t)$, with respect to the curve Γ, i.e.,

$$c_j := \frac{1}{2\pi i}\int_{\Gamma} t^{-j-1}\Delta_-^{-1}(t)A(t)dt, j = -æ,\ldots,0,\ldots,æ,$$

and to the degrees of polynomials

$$\mathscr{L}_{p+j}(t) = r_j(t), \quad \mathscr{R}_j(t) = t^{æ-\lambda_j+1}l_j(t), \quad j = 1,\ldots,p.$$

In Rodriguez & Campos (2013) a new approach to obtain factorization of a polynomial matrix functions $G \in \mathscr{G}(\mathscr{C}(\mathbb{T}))^{n\times n}$ was proposed. It is based on the relation between the general solution of the homogeneous Riemann boundary value problem and a solution to a linear system of difference equations with constant coefficients.

Considering a polynomial matrix function G in the form

$$G(t) = G_k t^k + \ldots + G_1 t + G_0, \quad t \in \mathbb{T}, \tag{5..1}$$

where $G_s, s = 0,1,\ldots,k$, are constant matrices, the authors associate with this matrix a homogeneous Riemann boundary value problem

$$\phi^+(t) = G(t)\phi^-(t), \quad t \in \mathbb{T}. \tag{5..2}$$

It was shown that the pair of vector functions $\phi^{\pm}$ is the solution to (5..2) where

$$\phi^-(t) = \phi_0 + \sum_{j=1}^{\infty}\phi_j t^{-j},$$

iff the coefficients $\phi_j, j = 0,1,\ldots,$ of the above series satisfy the following (infinite) system of linear difference equations

$$G_k\phi_{j+k} + \ldots + G_1\phi_{j+1} + G_0\phi_j = 0, \quad j = 0,1,\ldots. \tag{5..3}$$

The system (5..3) is analyzed by using the Z-transform method. Upon finding the solution to (5..3) the authors obtain the general solution to boundary value problem (5..2). In order to construct the canonical family of solutions (and thus the corresponding solution of the factorization problem) the authors appeal to the procedure described in Muskhelishvili (1968).

6. Rational matrix functions

Two main methods are considered for the factorization of rational matrix functions. The first method, coming back to the works by Gakhov (see Gakhov (1950), Gakhov (1952) and also Muskhelishvili (1968), Vekua (1967)), is related to the corresponding homogeneous Riemann boundary value problem

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma. \tag{6..1}$$

In the case of a non-singular matrix function $G(t)$ (i.e. $\det G(t) \neq 0, \forall t \in \Gamma$) rationally continued in the whole complex plane we separate the poles and zeroes of $G(z)$ and present it in the form

$$G(z) = U(z) \cdot V(z),$$

where some entries of matrices $U(z)$, $V(z)$ have (a finite number of) zeroes and poles in the domains D^-, D^+, and the determinants $\det U(z)$, $\det V(z)$ can be equal to zero or infinity in these domains, respectively.

Gakhov's idea was to find a polynomial matrix $L(z)$ with constant determinant, such that the matrix functions $\left(L(z)U^{-1}(z)\right)^{-1}$, $L(z)V(z)$ are analytic in D^-, D^+, respectively, and their determinants have no zeroes in the finite part of D^-, and in D^+, respectively. Moreover, the matrix $L(z)$ can be chosen in such a way, that the order of the determinant $\det\left(L(z)U^{-1}(z)\right)^{-1}$ at infinity is equal to the sum of the orders of the rows of this matrix (the order of a row at infinity is the smallest order of $\frac{1}{z}$ at infinity).[7]

Once found, the matrix functions

$$\tilde{G}^-(t) := \left(L(t)U^{-1}(t)\right)^{-1}, \quad \tilde{G}^+(t) := L(t)V(t)$$

satisfy the relation

$$\tilde{G}^-(t)\tilde{G}^+(t) = G(t).$$

It now remains to separate the diagonal matrix $\Lambda(t)$ and obtain the final factorization formula

$$G(t) = G^-(t)\Lambda(t)G^+, \tag{6..2}$$

where

$$\Lambda(t) = diag\left\{t^{-\kappa_1}, \ldots, t^{-\kappa_n}\right\},$$

and $\kappa_1, \ldots, \kappa_n$ are the orders of the rows of $\tilde{G}^-(t)$ at infinity,

$$G^-(t) = \tilde{G}^-(t)\left(\Lambda(t)\right)^{-1}, \; G^+ = \tilde{G}^+(t).$$

Therefore, the partial indices æ$_j$ are equal to $-\kappa_j$.

[7] Thus the zero of an entry at infinity has a positive order, and the pole has a negative. This is an original terminology by Gakhov (see Gakhov (1952)). Sometimes an opposite convention is used (see Muskhelishvili (1968), Vekua (1967)).

In the articles Gakhov (1950), Gakhov (1952), the finite algorithm for construction of the matrix $L(z)$ eliminates "bad points" (poles of the entries and poles/zeros of the determinants) of components U^{-1}, V of the given matrix G and makes their rows "canonical at infinity" (see Sec. 4.). It should be noted that this approach has to be slightly changed when contour Γ crosses the infinite point (in particular, when $\Gamma = \mathbb{R}$) due to the fact that in this case components of factorization G^-, G^+ play a more symmetric role.

Another approach to the factorization of a rational matrix function is by using the machinery of linear algebra. In Adukov (1991), Adukov (1993) the Wiener-Hopf factorization problem of a meromorphic matrix (i.e. a matrix rationally extended from the contour onto the whole complex plane) is reduced to the investigation of a finite system of linear homogeneous algebraic equations with matrix coefficients written in an explicit form. In particular, the exact formulas for partial indices are given in terms of the ranks of these matrix coefficients. The main result has the following form (see corresponding notation in Sec. 5.).

The right partial indices $\rho_1, \dots, \rho_n$, $\rho_1 \leqslant \dots \leqslant \rho_n$, and left partial indices $\lambda_1, \dots, \lambda_n$, $\lambda_1 \geqslant \dots \geqslant \lambda_n$, of the right- and left-factorization of the meromorphic matrix function $G(t)$, having N poles at points $t_1, \dots, t_N \in D^+$ of multiplicity $k_1, \dots, k_N$, respectively, are calculated by the following formulas

$$\rho_j = card\,\{k|p + r_{-k-1} - r_{-k} \leqslant j-1,\ k = 2\chi, 2\chi-1, \dots, 0\} - N - 1, \qquad (6..3)$$

$$\lambda_j = 2\chi - N + 1 - card\,\{k|r_{-k-1} - r_{-k} \leqslant j-1,\ k = 2\chi, 2\chi-1, \dots, 0\}, \qquad (6..4)$$

for $j = 1, \dots, p$. Here $card\,\{B\}$ is the number of elements in B, $\chi = ind_\Gamma \det G(t) + Np$, $r_{-2\chi-1} = 0$, r_{-k} are ranks of the matrices

$$T_{-k} = \frac{1}{2\pi i} \int_\Gamma \begin{pmatrix} t^k I \\ t^{k-1} I \\ \dots \\ t^0 I \end{pmatrix} G^{-1}(t) \left(t^0 I, t^1 I, \dots, t^{2\chi-k} I \right) \frac{dt}{t(t-t_1)^{k_1} \dots (t-t_n)^{k_n}}, \qquad (6..5)$$

where $k = 2\chi, 2\chi-1, \dots, 0$, and I is the unit matrix of order p.

The idea to use the ranks of the above matrices is similar to that used in Gohberg *et al.* (1980) (see also Ball & Clancey (1990)).

The factorization procedure of Adukov (1991), Adukov (1993) is applied in Adukov & Patrushev (2010) to create the algorithm employed to find the exact solution of the four-element generalized Riemann boundary value problem with rational coefficients on the unit circle. The algorithm is realized in the form of Maple routine.

In Adukov (2009) this procedure is generalized to the case of piece-wise meromorphic matrix functions (i.e. when D^+ is a multiply connected domain encircled by a finite number of the smooth closed curves).

In the case of generalized factorization (factorization in L_p-setting) the corresponding formulas for partial indices are obtained in Amirjanyan & Kamalyan (2007).

A left canonical factorization theorem for rational matrix functions relative to the unit circle is presented in Frazho & Kaashoek (2012). The result is a time invariant version of a recently proved the strict LU factorization theorem for certain semi-separable operators, due to Dewilde (2012) (see also Bart *et al.* (2008) and references therein). Explicit formulas for the factors are given too.

7. Symmetric matrix functions

It is known (see, e.g., Gohberg & Krein (1958a)) that any nonsingular Hölder continuous positive definite matrix function given on the real line or on the unit circle admits a canonical factorization. Moreover, in the case $\Gamma = \mathbb{T}$ or $\Gamma = \mathbb{R}$ the positive definiteness is a necessary and sufficient condition for a nonsingular Hölder continuous matrix function to be factorized in the form

$$G(t) = G^{+}(t)\left(G^{+}(t)\right)^{*}. \tag{7..1}$$

Taking this into account, as well as the polar decomposition of an arbitrary invertible matrix

$$G(t) = S(t)U(t),$$

where $S(t)$ is positive definite matrix and $U(t)$ is unitary matrix, the problem of determination of partial indices of an unitary matrix function given on the unit circle is discussed in Janashia & Lagvilava (1997). In particular, it was shown that a unitary matrix $U(t)$, $\det U(t) \equiv 1$

$$u_{ij}(t) = a_{ij}^{+}(t),\ u_{nj}(t) = \overline{a_{nj}^{+}(t)},\ 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n,$$

(with polynomials a_{ij}^{+}) admits a canonical factorization iff

$$\sum_{j=1}^{n} |a_{nj}^{+}(0)|^2 = 0.$$

In the case $n = 2$ the partial indices are calculated via the orders m_{2j} of zeros of $a_{2j}(z)$ at $z = 0$:

$$æ_1 = \min\{m_{21}, m_{22}\},\ æ_2 = -æ_1.$$

Discussion over the existence of factorizations of the type (7..1) date back to the work of Wiener (see Wiener (1955)). He showed that the sufficient condition for such a factorization of a positive definite matrix, with integrable entries given on the unit circle, has the following form

$$\log\det G(t) \in L_1(\mathbb{T}). \tag{7..2}$$

Later it was proved that condition (7..2) (called the Wiener-Paley condition) is the necessary one.

The coefficients of the analytic functions in the factor G^{+} are important for many applications, in particular, for prediction theory for stationary stochastic processes.[8] Some of the methods of approximate calculation (under certain additional conditions) were described in Masani (1960), Wiener & Masani (1958). In Janashia & Lagvilava (1999), a new effective approximation algorithm for the determination of the above coefficients was proposed, in the case of 2×2 matrices, where the only condition used was (7..2).

In Ephremidze *et al.* (2011) this approach is applied to the positive definite matrices of arbitrary order (see also Ephremidze *et al.* (2004), Janashia *et al.* (2011) and Ephremidze *et al.* (2015)). The proposed algorithm consists of several steps. First, any positive definite matrix G satisfying the Wiener-Paley condition (7..2) was represented in the form of a product of a lower triangular matrix $M \in L_2(\mathbb{T})$ and its conjugate. Second, the matrix $M(t)$ was approximated by the sequence of matrices $M_N(t)$ in $L_2(\mathbb{T})$ keeping only finite number of negative indices in the Fourier expansion of the under-diagonal entries of $M(t)$. Third, the spectral factor $S_N^{+}(t)$ of $S_N(t) = M_N(t)M_N^{*}(t)$ was explicitly computed. Fourth, the

[8] See the pioneering paper by Kolmogorov: Kolmogorov, A.N. Stationary sequences in Hilbert spaces, *Vestn. Mosk. Gos. Univ.*, No. 2, 1–40 (1941) (in Russian).

sequence of canonical spectral factors $\left(S_N^+\right)_c(z) = S_N^+(z)\left(S_N^+(0)\right)^{-1}\sqrt{S_N^+(0)\left(S_N^+(0)\right)^*}$ was determined, and it was shown that $\left(S_N^+\right)_c(z)$ converge to $(S^+)_c(z)$ in H_2.

A similar approach to that described above presented in Foster *et al.* (2010) (see also McWhirter *et al.* (2007)) where an algorithm for calculation of the so called *QR*-factorization and singular value decomposition of polynomial matrix function is proposed based on the Gram-Schmidt decomposition. The *QR*-factorization of a matrix A is its representation in the form $A = QR$, where Q ia an orthogonal matrix (for real valued A) or unitary matrix (for complex valued A), and R is an upper- (or a lower-) triangular matrix.

There is a long history of interest in construction of the factorization under certain symmetry conditions. We mention here the article Shmulian (1954), in which a method for an effective factorization of the Hermitian matrix on the unit circle was proposed. In Nikolajchuk & Spitkovskij (1975) it was shown that any self-adjoint (i.e. $G^*(t) = G(t)$) matrix function on the unit circle possesses so-called *self-adjoint factorization*, i.e.

$$G(t) = \left(G^+(t)\right)^* D_0(t) G^-(t),$$

where $D_0(t)$ is a block-diagonal matrix

$$D_0(t) = diag\left\{t^{-\alpha_1} I_{n_1}, \ldots, t^{-\alpha_r} I_{n_r}, \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}, t^{\alpha_r} I_{n_r}, \ldots, t^{\alpha_1} I_{n_1}\right\}.$$

In Krupnik *et al.* (1996) it was proved that any dissipative continuous matrix function of the form $A(t) = (t - z_0)^{-1} A_0 + B_+(t)$ $(t \in \Gamma)$, $z_0 \in D^+$, where A_0 is a constant matrix and $B_+(z)$ is analytic in D^+, admits a canonical factorization. Also, it was shown that for any non-simple contour Γ there exist 2×2 rational dissipative matrix functions and 2×2 Hölder continuous positive matrix functions which admit non-canonical factorization.

The canonical factorization of a rational matrix function $W(\lambda)$ which is analytic but may be not invertible at infinity is the subject of Gohberg & Zucker (1996). The factors were obtained explicitly in terms of the realization of the original matrix function. The cases of the symmetric factorization for self-adjoint and positive rational matrix functions are considered separately.

Some classes of continuous matrix functions with extra symmetry properties were studied in Voronin (2011).

8. Piece-wise constant matrix functions and the Riemann-Hilbert problem

The vector-matrix Riemann boundary value problem with piece-wise continuous algebraic coefficients was first solved by Hilbert (1912) by using Green's function method and later by Plemelj by reduction to a system of Fredholm integral equations, for which he proved existence of meromorphic solutions (which is of great relevance to the the Riemann-Hilbert problem). It was thought that Plemelj had found a complete and positive answer to the question of existence of the complex differential equation with a given monodromy group. However, in the late 1980s Bolibrukh (see e.g. Bolibrukh (1990)) showed that the proof of Plemelj is incomplete and that the negative answer is also possible. In Muskhelishvili & Vekua (1943) the authors studied problem (4..1) from the point of view of its application to the system of singular integral equations, considering only analytic solutions for (4..1), effectively picking up from where Plemelj had ceased.

The paper Ehrhardt & Spitkovsky (2001) is devoted to the connection between the factorization of piece-wise constant $n \times n$ matrix functions with m jumps and the Riemann-Hilbert problem. In studying these related problems, some results for the partial indices for general n and m were obtained, including complete answers for $n=2$, $m=4$ and for $n=m=3$. In some cases, the partial indices can be determined explicitly, while in the remaining cases, there remain two possibilities. The determination of the correct possibility is equivalent to the description of the monodromy of n-th order linear Fuchsian differential equations with m singular points. The simplest non-trivial case (n = 2, m = 3) was first studied by Zverovich & Khvoschinskaya (1985) in a slightly different setting. The problem was reduced to the boundary value problem on the Riemann surface, which was analyzed by using the Abel-type differentials of this surface.

Consider the system of linear differential equations in the complex domain

$$\frac{dy}{dz} = A(z)y. \tag{8..1}$$

Let $A(z)$ be a given $n\times n$ analytic matrix in $\mathbb{C}\setminus\{a_1,\dots,a_m\}$ (m singular points), $\tilde{S}$ be a universal covering manifold of $S=\mathbb{C}\setminus\{a_1,\dots,a_m\}$, $\rho:\tilde{S}\to S$, $\triangle$ be a group of covering automorphisms $\sigma:\tilde{S}\to\tilde{S}$, $\rho\circ\sigma=\rho$. Let $Y(\tilde{z})=(y_1(\tilde{z}),\dots,y_1(\tilde{z}))$, $\tilde{z}\in\tilde{S}$, be an analytic matrix, consisting of n linear independent solutions to (8..1), i.e. the solution vectors to the system

$$Y'(\tilde{z}) = A(z)Y(\tilde{z}). \tag{8..2}$$

There then exists a unique representation (matrix) $\chi(\sigma)$ such that

$$Y(\tilde{z}) = Y(\sigma(\tilde{z}))\chi(\sigma).$$

A class of mutually conjugated representations is called the *monodromy* of system (8..2) (or of system (8..1)). This class is generated by matrices $M_1,\dots,M_m$, $M_1\cdot\ldots\cdot M_m=I$, and is denoted $[M_1,\dots,M_m]_\sim$. Locally, $Y(\tilde{z})=Z_k(z)(\tilde{z}-a_k)^{E_k}$ at a neighborhood of $z=a_k$, where $M_k\sim\exp(-2\pi iE_k)$, $\forall k=1,\dots,m$, and "$\sim$" stands for similarity of matrices.

A singular point a_k is called *Fuchsian* if $A(z)$ has at $z=a_k$ only a simple pole. A singular point a_k is called *regular* if the fundamental matrix $Y(z)$ does not have an essential singularity at $z=a_k$.

For any mutually disjoint points $a_1,\dots,a_m\in\mathbb{C}$ and matrices $M_1,\dots,M_m\in\mathbb{C}^{n\times n}$, $M_1\cdot\ldots\cdot M_m=I$, there exists the system (8..1) with regular singularities only at $a_1,\dots,a_m$ and with a monodromy of the class $[M_1,\dots,M_m]_\sim$ (Plemelj[9]).

Plemelj's result is, in fact, more general: There exists a system with all Fuchsian points, except possibly one at which the singularity is not higher than regular.

Question: Does there exist a system with prescribed Fuchsian singularities $a_1,\dots,a_m$ and a given monodromy? This is called the *Riemann-Hilbert problem* or the *21-st Hilbert problem*, to which Bolibrukh (2009) has provided a negative answer.

Let $a_1,\dots,a_m$ be situated on a closed smooth curve Γ. A matrix E is called *non-resonant* if none of the differences between its eigenvalues equals an entire number. $[M_1,\dots,M_m]$, $[E_1,\dots,E_m]$ are called *admissible data* if

(a) $M_1\cdot\ldots\cdot M_m=I$,

(b) $M_k\sim\exp(-2\pi iE_k)$, $\forall k=1,\dots,m$,

(c) matrices $E_1,\dots,E_m$ are non-resonant.

The system (8..2) with singular points $a_1,\dots,a_m,\infty$, and indexes $æ_1,\dots,æ_n$ is of *a standard form* with respect to admissible data $[M_1,\dots,M_m]$, $[E_1,\dots,E_m]$, if ∞ is a removable singularity, and $Y(\tilde{z})=$

[9] Plemelj, J. *Problems in sense of Riemann and Klein*, Tracts in Pure and Appl. Math., **16**, New York: J. Wiley and Sons, 1964.

$Z_k(z)(\tilde{z}-a_k)^{E_k}C$ at a neighborhood of $z=a_k$, $Y(z)=\mathrm{diag}\,(z^{æ_1},\ldots,z^{æ_n})\,Z_\infty(z)C$ in a neighborhood of infinity (Ehrhardt & Spitkovsky (2001)).

Let $G\in PC\,(\Gamma)^{n\times n}$ be a piece-wise constant matrix with jumps only at the points $a_1,\ldots,a_m$. We assume that G admits a Φ-factorization in $L^p\,(\Gamma)\,, 1<p<\infty$, and $[M_1,\ldots,M_m]$, $[E_1,\ldots,E_m]$ are corresponding data.

If there exists a system of a standard form with singular points $a_1,\ldots,a_m$, ∞, indexes $æ_1,\ldots,æ_n$, and Y_1 and Y_2 are solutions of this system in D_+ and $D_-\setminus\{\infty\}$, respectively, then there are constant nonsingular matrices C_1, C_2, such that the pair $G_+(z)$, $G_-(z)$

$$G_+(z)=C_1^{-1}Y_1^{-1}(z), z\in D_+;\quad G_-(z)=\Lambda^{-1}(z)Y_2(z)C_2, z\in D_-\setminus\{\infty\}.$$

generates the Φ-factorization of the matrix G in $L^p(\Gamma)$:

$$G(t)=G_+(t)\Lambda(t)G_-(t), t\in\Gamma,$$

where $\Lambda(t)=\mathrm{diag}\,\{t^{æ_1},\ldots,t^{æ_n}\}$.

In this setting, which was considered in Ehrhardt & Spitkovsky (2001), the problem was partly resolved by Spitkovskij & Tashbaev (1991). Using the appropriate modification of the results from Zverovich & Khvoschinskaya (1985), they gave explicit formulas for the factors and the partial indices. In the paper, they had to distinguish several cases, wherein, apart from the trivial cases (where G can be reduced to a functionally commuting matrix function), the factors were constructed by using the hypergeometric functions.

9. Daniele-Khrapkov approach

The *Daniele-Khrapkov matrix functions* are 2×2 matrix functions which can be written in the form

$$G(t)=a(t)I+b(t)R(t),\quad t\in\mathbb{R}, \tag{9..1}$$

where a and b are scalar functions, I is the unit matrix and R is a polynomial 2×2 matrix function whose trace is zero. The polynomial $\sigma(t)=-\det R(t)$ is called the *deviator polynomial* and contains important information with regard to the factorization of G. The study of Daniele-Khrapkov matrix functions is based on the idea of the commutative matrix factorization in certain subalgebras, and began with the works of Khrapkov (see Khrapkov (1971a), Khrapkov (1971b)), and of Daniele (1978). This work continued with the explicit solution of canonical diffraction problems (see, e.g., the literature cited in the survey paper Meister & Speck (1989)). Later, Daniele-Khrapkov matrix functions were systematically considered (see, e.g. the articles described below in this section and references therein), but the general case of the factorization of the Daniele-Khrapkov matrix functions with deviator polynomial of arbitrary degree is still open (see, Prössdorf & Speck (1990), and further discussion in Ehrhardt & Speck (2002)).

A characterization of the Daniel-Khrapkov matrix functions is given in Ehrhardt & Speck (2002) in terms of linear independence with respect to certain field K of an infinite characteristic (a classical example of K is the field of rational functions restricted to Γ, i.e. $\mathrm{K}=\mathscr{R}(\Gamma)$). Let $\mathscr{B}^{2\times 2}$ be an algebra of 2×2 matrix-functions with entries from K (e.g. $\mathscr{B}^{2\times 2}=\mathscr{G}\,(\mathscr{R}(\Gamma))^{2\times 2}$). Then, by nr_{K} we denote the number of independent entries of $A\in\mathscr{B}^{2\times 2}$ over the field K:

$$nr_{\mathrm{K}}(A)=\dim\,\mathrm{lin}_{\mathrm{K}}\,\{a_{11},a_{12},a_{21},a_{22}\}\,. \tag{9..2}$$

It has been proved (see Ehrhardt & Speck (2002), cf. also Prössdorf & Speck (1990)), that a matrix $A\in\mathscr{B}^{2\times 2}$ is of the Daniel-Khrapkov type iff one of the following conditions is satisfied:

(a) $nr_{\mathrm{K}}(A) \leqslant 1$,
(b) $nr_{\mathrm{K}}(A) = 2$ and $I \in Z_{\mathrm{K}}(A)$, where

$$Z_{\mathrm{K}}(A) = \left\{Q \in \mathrm{K}^{2\times 2} : trace(SQ) = 0, \forall S \in \mathrm{K}^{2\times 2},\, trace(SA) = 0\right\}.$$

In Ehrhardt & Speck (2002) the aim was to describe classes of those invertible matrices and of those transformations of the type (3..1), for which the transformed matrix $\hat{G}$ could be either triangular or of the Daniele-Krapkov form. The systematic study of rational transformations into Daniele-Khrapkov matrix functions in factorization theory begun by Prössdorf & Speck (1990). It was proved, for example, that an invertible 2×2 matrix function (with entries belonging to the Wiener algebra defined on the unit circle) can be transformed by an invertible rational transformation (having no pole on Γ) into a Daniele-Khrapkov matrix function if and only if the matrix function has at most two rational independent entries. For the transformations of the type (3..2) the situation is slightly different: an invertible matrix function G can be transformed by a transformation of the type (3..2) into a Daniele-Khrapkov matrix function if and only if G has at most two rationally independent entries – apart from the exceptional case. In this exceptional case, a transformation into a Daniele-Khrapkov matrix function may or may not be possible, however, in this exceptional case a transformation into a triangular matrix function is always possible.

The Daniele-Khrapkov type matrix can be written in the form (see Abrahams (1998))

$$G(t) = I + f(t)J(t), \tag{9..3}$$

where $f(t)$ is an arbitrary scalar function of t having algebraic behaviour at infinity, $J(t)$ is a square matrix with polynomial entries such that

$$J^2(t) = \Delta^2(t)I, \tag{9..4}$$

and thus $\Delta^2(t)$ is a polynomial in t. A commutative product factorization of $G(t)$ is the representation of the form

$$G(t) = Q_-(t)Q_+(t), \tag{9..5}$$

where $Q_\pm$ and their inverses possess an analytic continuation in the domains $\Pi_\pm := \{z : \pm \mathrm{Im} z > 0\}$. It is known that in the case of the Daniele-Khrapkov matrices the factors can be represented as

$$Q_\pm(t) = r_\pm(t)\left\{\cosh[\Delta(t)\theta_\pm(t)]I + \frac{1}{\Delta(t)}\sinh[\Delta(t)\theta_\pm(t)]J(t)\right\}. \tag{9..6}$$

Note that $\Delta(t)$ has branch-points in both half-planes, in general, but this does not affect $Q_\pm(t)$ because they are in fact functions of $\Delta^2(t)$.

Thus the factorization problem is reduced to the problem of determination of the scalar functions $r_\pm(t), \theta_\pm(t)$. It has been shown that they satisfy the equations

$$r_+(t)r_-(t)\cosh[\Delta(t)(\theta_+(t) + \theta_-(t))] = 1, \tag{9..7}$$

$$\frac{r_+(t)r_-(t)}{\Delta(t)}\sinh[\Delta(t)(\theta_+(t) + \theta_-(t))] = f(t), \tag{9..8}$$

or in more standard form

$$(r_+(t)r_-(t))^2 = 1 - \Delta^2(t)f^2(t), \tag{9..9}$$

$$\theta_+(t) + \theta_-(t) = \frac{1}{\Delta(t)}\tanh^{-1}[\Delta(t)f(t)]. \tag{9..10}$$

The solutions of these problems can be written explicitly using Gakhov's formulas.

In Câmara *et al.* (1995a), Câmara *et al.* (1995b) the factorization problem related to the Daniele-Khrapkov approach is considered in the following setting. Let $G : \mathbb{R} \to \mathbb{C}^{2\times 2}$ be of the following form

$$G = \begin{pmatrix} a & b \\ \rho^2 b & a \end{pmatrix},$$

where $a, b \in \mathscr{C}(\dot{\mathbb{R}})$, and $\rho \in \mathscr{C}(\dot{\mathbb{R}})$ is the square root of the quotient of two second degree polynomials, such that $a \pm \rho b$ possess a bounded generalized (Simonenko-type) factorization related to $L_p(\mathrm{R}), p > 1$. In Câmara *et al.* (1995a) the necessary and sufficient conditions for canonical factorization were found, as were explicit formulas for the partial indices in the non-canonical case Câmara *et al.* (1995a). In Câmara *et al.* (1995b) explicit formulas for the canonical factorization were derived. As a by-product of these investigations, a solution was given to the problem posed by Daniele, specifically that of determining of a rational matrix function R in the same group as G, such that the factors in factorization of RG belongs to this group too.

In Câmara & Malheiro (2000), the authors generalized the Daniele-Khrapkov approach, by using generalized factorization of matrices in the form

$$G = \alpha I + \beta N, \quad N = \begin{pmatrix} 1 & q \\ -q^{-1} & -1 \end{pmatrix},$$

where q is a non-zero rational function without poles on $\mathbb{R}$, $\alpha, \beta \in \mathscr{C}\left(\dot{\mathbb{R}}\right)$. This class of matrices is related to the group of matrices of the form $I + \gamma N$, where $\gamma \in \mathscr{C}\left(\dot{\mathbb{R}}\right)$ and N is a rational nilpotent matrix. For such matrices the necessary and sufficient conditions for the existence of canonical generalized factorization and canonical factorization with factors in the same group, as well as explicit formulas for the factors, were determined. Non-canonical factorization was also studied. This class of matrix functions presents some interesting characteristics which distinguish it from the Daniele-Khrapkov class, in spite of the formal similarity. In particular, the conditions for existence of a canonical factorization are quite different.

Developing the Wiener-Hopf technique, in Jones (1984) (see also Moiseev (1989), Moiseev (1993)), the Daniele-Khrapkov class was generalized, specifically, to $n \times n$ matrices of the form

$$G = g_0 I + g_1 J + \ldots + g_{n-1} J^{n-1} \tag{9..11}$$

were investigated. Here g_k are arbitrary scalar functions of the complex variable, analytic in the Wiener-Hopf strip $\mathscr{D}$ with algebraic growth at infinity, J is an entire matrix with polynomial entries such that

$$J^n = \Delta^n I, \tag{9..12}$$

and analogously to Khrapkov case, Δ^n is a polynomial in complex variable, $\Delta^n(z) = O(z^p)$ as $z \to \infty$ (with p greater than some constant). Key ingredients of the method in Jones (1984) were the following (see also Veitch & Abrahams (2007)).

The matrix G was rewritten in the form

$$G = \exp\left\{\sum b_i B_i\right\}, \tag{9..13}$$

where B_i have the orthogonality property

$$B_i B_j = \begin{cases} 0, & for \quad i \neq j \\ B_i, & for \quad i = j. \end{cases} \tag{9..14}$$

By the Cayley-Hamilton theorem n eigenvalues of J, which we denote by λ_i, satisfy $\lambda_i = \omega^i\Delta$, where $0 \leqslant i \leqslant n$ and $\omega^n = 1$ (excluding the possibility $\omega = 1$). Hence the matrices B_i are linear combinations of $\{I, J, \ldots, J^{n-1}\}$ and are presented by the following formula:

$$B_i = \frac{1}{n}\sum_{r=0}^{n-1}\frac{\omega^{ir}}{\Delta^r}J^r, \tag{9..15}$$

$$\sum_{i=0}^{n-1}\omega^{ip} = \begin{cases} 0, & p \equiv 1, 2, \ldots, n-1, \quad mod\, n \\ n, & p \equiv 0, \quad mod\, n. \end{cases} \tag{9..16}$$

A commutative Wiener-Hopf factorization of the matrix G

$$G(t) = G^-(t)G^+(t) \tag{9..17}$$

is then (see Jones (1984), cf. Veitch & Abrahams (2007)) given by

$$G^{\pm} = \exp\left\{\sum_{i=0}^{n-1} l_i^{\pm} J^i\right\} \tag{9..18}$$

where $l_i^{\pm}$ represents the Cauchy sum split of the functions l_i which are given by

$$l_i = \frac{1}{n\Delta^i}\sum_{j=0}^{n-1}\omega^{ij}\log\left\{\sum_{r=0}^{n-1}\omega^{-jr}\Delta^r g_r\right\}. \tag{9..19}$$

Motivated by the results of Jones (1984), a method for factorizing $n \times n$ matrix functions (Wiener-Hopf kernels) with $n > 2$ and the commuting factors was presented in Veitch & Abrahams (2007). The proposed technique is supposed to be applicable to the studying problems of mechanics and mathematical physics, and is illustrated by consideration of 3×3 matrix functions arising from elastostatic theory.

10. Non-rational matrix functions

An explicit factorization of a class of non-rational 2×2 matrix functions was obtained in Aktosun *et al.* (1992). These matrices are related to the Wiener-Hopf problem and also appear as modified scattering matrices for a variant of the 1-dimensional Schrödinger equation.

The authors considered the matrix-functions of the type

$$G(k,x) = \begin{pmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{2ikx} & T(k) \end{pmatrix}, \quad k \in \Gamma = \mathbb{R}, \tag{10..1}$$

dependent on the real parameter x under the following conditions:

1) $T(k) \neq 0$ in the "closed" upper half-plane apart from the origin (i.e. $\forall k \in \{cl\,\Pi^+\} \setminus \{0\} = \{k \in \mathbb{C} : \mathrm{Im}\, k \geqslant 0\} \setminus \{0\}$), is meromorphic in Π^+ with continuous boundary values on $\overline{\mathbb{R}}$; either $T(0) \neq 0$ or the order of zero of $T(k)$ at $k = 0$ is finite; $T(\infty) = 1$;

2) $R(k)$ and $L(k)$ are meromorphic on Π^+ with continuous boundary values on the extended real axis and vanish as $cl\,\Pi^+ \ni k \to \infty$;

3) $G(k,x)^{-1} = \mathbf{q}\,G(-k,x)\,\mathbf{q}$ for $k \in \mathbb{R}$, where $\mathbf{q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

4. $G(k,x)$, as a function of $k \in \mathbb{R}$, belongs to a suitable Banach algebra of 2×2 matrix functions within which the Wiener-Hopf factorization is possible.

The main idea is to relate the factorization problem to the two auxiliary Riemann boundary value problems (with a flip[10])

$$\mathbf{m}(-k,x) = \left(\frac{k+i}{k-i}\right)^{\tau} \cdot G(k,x) \cdot \mathbf{q} \cdot \mathbf{m}(k,x), \quad k \in \mathbb{R}, \tag{10..2}$$

$$\mathbf{n}(-k,x) = \left(\frac{k+i}{k-i}\right)^{\sigma} \mathbf{J} \cdot G(k,x) \cdot \mathbf{J} \cdot \mathbf{q} \cdot \mathbf{n}(k,x), \quad k \in \mathbb{R}, \tag{10..3}$$

where $\mathbf{J} = diag\,\{1,-1\}$, and the parameters τ, σ are uniquely determined by the coefficients of the above problems.

It was shown that whenever solutions of the problems (10..2), (10..3) are constructed, then the factorization of the matrix $G(k,x)$ is found explicitly in the following special form

$$G(k,x) = G_{-}(k,x)\left[\left(\frac{k-i}{k+i}\right)^{\tau} \mathbf{Q}_{+} + \left(\frac{k-i}{k+i}\right)^{\sigma} \mathbf{Q}_{-}\right] G_{+}(k,x), \tag{10..4}$$

where $\mathbf{Q}_{\pm}$ can be chosen as $\mathbf{Q}_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$, and $G_{-}(k,x)$, $G_{+}(k,x)$ are just certain combinations of the components of the solutions to (10..2), (10..3).

In Feldman *et al.* (1994), the authors proposed an algorithm for the explicit factorization of 2×2 matrix functions

$$G(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad |t| = 1, \tag{10..5}$$

with entries from the Wiener algebra $\mathscr{W} = \mathscr{W}(\mathbb{T})$ of absolutely convergent Fourier series. This algorithm was realized in Feldman *et al.* (1994), in the case when at least one of the following conditions holds:

0.1) the function b/a admits a meromorphic extension into D^{+} (the interior of the unit disc $|t| < 1$);

0.2) the function c/a admits a meromorphic extension into D^{-} (the exterior of the unit disc $|t| > 1$).

The algorithm consists of the solution of two scalar homogeneous Riemann boundary value problems and of a finite system of the linear equations.

The algorithm is applied in Feldman *et al.* (1994) to solve various classes of singular integral equations and equations with Toeplitz and Hankel matrices. Based on this algorithm, the paper Feldman *et al.* (1995) was devoted to two topics connected with factorization of triangular 2×2 matrix functions. The first application is an explicit factorization of a class of matrices of Daniel-Khrapkov type and the second is related to inversion of the finite Toeplitz matrices.

A slight extension of the conditions of Feldman *et al.* (1994) is given in Feldman *et al.* (2004), where application of the algorithm to two types of matrices from $\mathscr{W}^{2\times 2}$ was considered. The first of these types is

$$G(t) = \begin{pmatrix} 1 & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad |t| = 1, \tag{10..6}$$

[10] See, e.g., Litvinchuk (2000).

where $b(t) = p(t)/q(t)$, $p(t)$ is analytically continued into D^+, i.e. $p(t) \in W^+$, and

$$q(t) = \prod_{j=1}^{k}(t-\alpha_j)^{m_j}, \quad |\alpha_j| < 1.$$

The corresponding scalar boundary value problems are given in an explicit form and the system of equations presented as $u^{(r)}(\alpha_j) = 0, r = 0, 1, \ldots, m_j,$, where $u(t)$ is a combination of the components of the solutions to the above boundary value problem.

The second type of studied matrices is

$$G(t) = \begin{pmatrix} 1 & b(t) \\ -1/b(t) & 1 \end{pmatrix}, \quad |t| = 1, \tag{10..7}$$

where $b \in \mathscr{W}$, b^2 is a rational function without zeros and poles on $\mathbb{T}$, $b = w_- w_+$, where $w_\pm$ admits a meromorphic extension into $D^\pm$, and w_-^2 is a rational function.

A number of other approaches have been proposed to factorize non-rational matrix functions, especially those related to certain problems appearing in applications.

In Abrahams (1997) the coupled Wiener-Hopf equations were considered with matrix coefficients of the form

$$K(\alpha) = \begin{pmatrix} 1 & \mu\gamma(\alpha) \\ -\mu/\delta(\alpha) & 1 \end{pmatrix},$$

where the aim of the article was to demonstrate a new procedure for obtaining noncommutative matrix factors which have algebraic growth. Padé approximants were employed to obtain an approximate but explicit noncommutative factorization of the matrix kernel. As well as being simple in application, the approximants allows increased accuracy of the factorization.

In Câmara & dos Santos (2000) a class of 2×2 matrix functions was studied in a generalized setting (with respect to $L^2(\mathbb{R})$). The considered class was motivated by an inverse-scattering problem and by the theory of convolution operators on the finite interval, and consists of matrices of the type

$$G(t) = \begin{pmatrix} a(t) & b(t)e^{it} \\ b(t)e^{-it} & a(t) \end{pmatrix}, \quad a, b \in L^\infty(\mathbb{R}). \tag{10..8}$$

Supposing that the functions $a \pm b$ admit bounded canonical factorization the authors demonstrated the possibility of representing the matrix G in the form of the product of three triangular matrix functions. Under some additional conditions, an explicit generalized canonical factorization of G was obtained. The proposed method was based on an analysis of solvability of a series of the corresponding boundary value problems.

In Mishuris & Rogosin (2014), an asymptotic method for canonical factorization of a class of matrix functions of arbitrary order defined on the real line was proposed. This choice of matrices was motivated by certain problems based in the theory of elasticity. An example is constructed to illustrate the efficiency of the proposed procedure, and the quality of approximation and the role of the chosen small parameter was discussed.

11. Discussion and further study

In this survey we have collected results mainly related to the constructive factorization of matrix functions in classical setting. A few related definitions (generalized Simonenko type factorization, spectral factorization, *QR*-factorization, symmetric and antisymmetric factorization, self-adjoint factorization) are mentioned only briefly. In our further study we plan to widen our focus to include other generalizations of classical methods. We will also consider some special factorization procedures which have been developed to solve specific problems in applied science.

We have here restricted our review to the results dealing only with simple geometries (the considered matrices are mainly defined either on the unit circle or on the real line). Despite the fact that this geometrical restriction allows formulation of results in a clear form, we have to note that from both the theoretical and applied points of view the study of the factorization problem as it relates to more general geometry is also of interest.

Few challenging questions have a specific answer. First, some of the above results are obtained only in the case of 2×2 matrices, as discussion of the arbitrary order matrices is either too cumbersome or even impossible. Much deeper understanding is therefore required in many cases. Second, further investigation of non-rational matrix functions is required, as the known factorizing procedures are developed only for certain classes of matrices. Third, the rectangular matrix functions have yet to be considered, and require the development of a unified approach. Certainly, such an approach will be not as simple as for square matrices.

A few things must be mentioned about the relationship between the discussed problems and matrix theory. There are several types of factorization of constant matrices, where the most well known are diagonalization, Jordan canonical decomposition (see e.g. Gantmacher (1967), Horn & Johnson (1985), Horn & Johnson (1991)). In many books on matrix theory (see e.g. Higham (2008) and references therein) the notion of matrix functions is also discussed, but it is rather functions of matrix than vice versa. A similarity clearly exists between these two theories (of functional matrices and of constant matrices). The most closed aspect of matrix theory to our survey is the so called eigendecomposition of constant matrices, i.e. the representation of a constant matrix A in the form $A = ZDZ^{-1}$, where $D = diag\{\lambda_j\}$, where the λ_j are the eigenvalues of A, and the columns of Z are the eigenvectors of A. The discussion of a unified point of view for both theories is also a challenging and open topic.

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References

ABRAHAMS, I. D., (1997) On the solution of Wiener-Hopf problems involving noncommutative matrix kernel decompositions, *SIAM J. Appl. Math.,* **57** (2), 541–567.

ABRAHAMS, I. D., (1998) On the non-commutative factorization of Wiener-Hopf kernels of Khrapkov type, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.,* **454**, 1719–1743.

ADUKOV, V.M., (1991) On Wiener-Hopf factorization of meromorphic matrix functions, *Integral Equations Operator Theory,* **14** (6), 767–774.

ADUKOV, V.M., (1993) Wiener-Hopf factorization of meromorphic matrix-functions, *St. Petersburg Math. J.,* **4** (1), 51–69.

ADUKOV, V.M., (1999) Factorization of analytic matrix-valued functions, *Theoret. and Math. Phys.,* **118** (3), 255–263.

ADUKOV, V.M., (2009) Wiener-Hopf factorization of piece-wise meromorphic matrix-valued functions, *Sb. Math.,* **200** (8), 1105–1126.

ADUKOV, V.M. & PATRUSHEV, A.A., (2010) Algorithm of exact solution of generalized four-element Riemann-Hilbert boundary value problem with rational coefficients and its programm realization, *Vestn. Yuzhno-Ural. Gos. Univ.* Ser. Mat. Model. Program. Vyp. 6, No. 35 (211), 4–11 (in Russian).

AKTOSUN, T., KLAUS, M. & VAN DER MEE, C., (1992) Explicit Wiener-Hopf factorization for certain non-rational matrix functions, *Integral Equations Operator Theory*, **15**, 879–900.

AMIRJANYAN, H.A. & KAMALYAN, A.G., (2007) Factorization of meromorphic matrix functions, *J. Contemp. Math. Anal., Armen. Acad. Sci.* **42** (6), 303–319; translation from *Izv. Nats. Akad. Nauk Armen., Mat.* No. 6, 1–24.

BABICH, V. M., (2009) Factorizability of matrix functions: A direct proof, *St. Petersburg Math. J.*, **20**, 1–22.

BALL, J.A. & CLANCEY, K.F., (1990) An elementary description of partial indices of rational matrix functions, *Integral Equations Operator Theory*, **13**, (3), 316–322.

BART, H., GOHBERG, I. & KAASHOEK, M.A., (1986) Explicit Wiener-Hopf factorization and realization, In: *Constructive methods of Wiener-Hopf factorization,* Operator Theory: Advances and Applications, **21**. Basel: Birkäuser, 317–355.

BART, H., GOHBERG, I., KAASHOEK, M.A. & RAN, A.C.M., (2008) *Factorization of Matrix and Operator Functions: The State Space Method.* Operator Theory: Advances and Applications, **178**. Basel: Birkäuser.

BHATIA, R., (1994) Matrix factorizations and their perturbations, *Linear Algebra Appl.,* **197**/**198**, 245–276.

BOJARSKY, B., (1958) On stability of the Hilbert problem for holomorphic vector, *Soobstch. AN GruzSSR*, **21** (4), 391–398 (in Russian).

BOLIBRUCH, A. A., (1990) The Riemann–Hilbert problem, *Russian Math. Surveys.,* **45** (2), 1–47.

BOLIBRUCH, A. A., (2009) *Inverse Monodromy Problems in the Analytic Theory of Differential Equations,* Moscow: MTsNMO (in Russian).

BÖTTCHER, A., KARLOVICH, YU I. & SPITKOVSKY, I.M., (2002) *Convolution operators and factorization of almost periodic matrix functions*, Operator Theory: Advances and Applications, **131**. Basel: Birkäuser.

BÖTTCHER, A. & SILBERMANN, B., (1990) *Analysis of Toeplitz Operators*. Berlin: Springer.

BÖTTCHER, A. & SPITKOVSKY, I.M., (2013) The factorization problem: some known results and open questions. In: A.ALMEIDA, L.CASTRO & F.-O.SPECK, eds. *Advances in Harmonic Analysis and Operator Theory,* Operator Theory: Advances and Applications, **229**. Basel: Birkäuser, 101–122.

CÂMARA, M. C., LEBRE, B. & SPECK, F.-O., (1992) Meromorphic factorization, partial index estimates and elastodynamic diffraction problems, *Math. Nachr.*, **157**, 291–317.

CÂMARA, M. C. & MALHEIRO, M.T., (2000) Wiener-Hopf factorization for a group of exponentials of nilpotent matrices, *Linear Algebra Appl.,* **320**, 79–96.

CÂMARA, M. C. & DOS SANTOS, A.F., (1994) Generalized factorization for a class of $n \times n$ matrix functions. - Partial indices and explicit formulas, *Integral Equations Operator Theory,* **20** (2), 198–230.

CÂMARA, M. C. & DOS SANTOS, A.F., (2000) Wiener-Hopf factorization for a class of oscillatory symbols, *Integral Equations Operator Theory,* **36** (4), 409–432.

CÂMARA, M.C.; DOS SANTOS, A.F. & BASTOS, M.A., (1995) Generalized factorization for Daniele-Khrapkov matrix functions - partial indices, *J. Math. Anal. Appl.* **190**, 142–164.

CÂMARA, M.C.; DOS SANTOS, A.F. & BASTOS, M.A., (1995) Generalized factorization for Daniele-Khrapkov matrix functions - explicit formulas, *J. Math. Anal. Appl.* **190**, 295–328.

CÂMARA, M.C.; DOS SANTOS, A.F. & MANOJLOVIC, N., (2001) Generalized factorization for $n \times n$ Daniele-Khrapkov matrix functions, *Math. Methods Appl. Sci.* **393**, 993–1020.

CASTRO, L.P., DUDUCHAVA, R. & SPECK, F.-O., (2005) *Asymmetric Factorizations of Matrix Functions on the Real Line*. Departamento de Matemática Instituto Superior Técnico, Lisboa, Portugal, Preprint 13/2005, 25 pp.

CHEBOTAREV, G. N., (1956) Partial indices of the Riemann boundary value problem with a triangular matrix of the second order, *Uspekhi Mat. Nauk,* **11** (3(69)), 192–202 (in Russian).

CHEBOTAREV, G. N., (1956) On the solution in an explicit form of the Riemann boundary value problem for a system of n pairs of functions, *Uch. zapiski Kazan State University. Mathematika.* **116** (4), 31–58 (in Russian).

CLANCEY, K. & GOHBERG, I., (1987) *Factorization of Matrix Functions and Singular Integral Operators*. Operator Theory: Advances and Applications, **25**. Basel: Birkäuser.

DANIELE, V.G., (1978) On the factorization of Wiener-Hopf matrices in problems solvable with Hurd's method, *IEEE Trans. Antennas and Propagation,* **26**, 614–616.

DANIELE, V.G., (1984) On the solution of two coupled Wiener-Hopf equations, *SIAM J. Appl. Math.* **44** (4), 667–680.

DANIELE, V.G., (2003) The Wiener-Hopf technique for impenetrable wedges having arbitrary aperture angle, *SIAM J. Appl. Math.* **63** (4), 1442–1460.

DEWILDE, P. & VANDEWALLE, T.P., (1975) On the factorization of a nonsingular rational matrix, *IEEE Trans. Circuits Systems I Fund. Theory Appl.* **22**, 637–645.

DEWILDE, P., (2012) On the LU factorization of infinite systems of semi-separable equations, *Indag. Math. (N.S.)* **23**, 1028–1052.

DUDUCHAVA, R. (1979) *Integral equations with fixed singularities*. Leipzig: BG Teubner.

DUDUCHAVA, R. & WENDLAND, W.L., (1995) The Wiener-Hopf method for systems of pseudodifferential equations with an application to crack problems, *Integral Equations Operator Theory,* **23** (3), 294–335.

EHRHARDT, T. & SPECK, F.-O., (2002) Transformation techniques towards the factorization of non-rational 2×2 matrix functions, *Linear Algebra Appl.,* **353** (1–3), 53–90.

EHRHARDT, T. & SPITKOVSKY, I.M., (2001) Factorization of piece-wise constant matrix-functions and systems of differential equations, *Algebra i Analiz,* **13** (6), 56–123; Engl. transl. in: *St. Petersbg. Math. J.,* **13** (6), 939–991.

EPHREMIDZE, L., JANASHIA, G. & LAGVILAVA, E., (1998) On the factorization of unitary matrix-functions, *Proc. A. Razmadze Math. Inst.,* **116**, 101–106.

EPHREMIDZE, L., JANASHIA, G. & LAGVILAVA, E., (2004) A new computational algorithm of spectral factorization for polynomial matrix-functions, *Proc. A. Razmadze Math. Inst.,* **136**, 41–46.

EPHREMIDZE, L., JANASHIA, G. & LAGVILAVA, E., (2008) An analytic proof of the matrix factorization theorem, *Georgian Math. J.* **15** (2), 241–249.

EPHREMIDZE, L., JANASHIA, G. & LAGVILAVA, E., (2011) An approximate spectral factorization of matrix functions, *J. Fourier Anal.* **17**, 976–990.

EPHREMIDZE, L., LAGVILAVA, E. & SPITKOVSKY, I. (2015) Rank-deficient spectral factorization and wavelets, *Int. J. Wavelets Multiresolut. Inf. Process.,* **13**, 1550013, DOI: 10.1142/S0219691315500137

FELDMAN, I., GOHBERG, I. & KRUPNIK, N., (1994) A method of explicit factorization of matrix functions and applications, *Integral Equations Operator Theory,* **18** (3), 277–302.

FELDMAN, I., GOHBERG, I. & KRUPNIK, N., (1995) On explicit factorization and applications, *Integral Equations Operator Theory,* **21**, 430–459.

FELDMAN, I., GOHBERG, I. & KRUPNIK, N., (2000) Convolution equations on finite intervals and factorization of matrix functions, *Integral Equations Operator Theory,* **36** (2), 201–211.

FELDMAN, I., GOHBERG, I. & KRUPNIK, N., (2004) An explicit factorization algorithm, *Integral Equations Operator Theory,* **49**, 149–164.

FELDMAN, I. & MARKUS, A., (1998) On some properties of factorization indices, *Integral Equations Operator Theory,* **30**, 326–337.

FRAZHO, A.E. & KAASHOEK, M.A., (2012) Canonical factorization of rational matrix functions. A note on a paper by P. Dewilde, *Indag. Math. (N.S.),* **23**, 1154–1164.

FOSTER, J.A., MCWHIRTER, J.G., DAVIES, M.R. & CHAMBERS, J.A., (2010) An algorithm for calculating the QR and singular value decomposition of polynomial matrices, *IEEE Trans. Signal Process.* **58** (3), 1263-1274.

GAKHOV, F.D., (1950) One case of the Riemann boundary value problem for a system of n pairs of functions, *Izv. AN SSSR Ser. Mat.,* **14**, 549–568 (in Russian).

GAKHOV, F.D., (1952) Riemann's boundary value problem for a system of n pairs of functions, *Uspekhi Mat. Nauk,* **VII** (4 (50)), 3–54 (in Russian).

GAKHOV, F.D., (1977) *Boundary Value Problems*, 3rd ed., Moscow: Nauka (1977) (in Russian).

GAKHOV, F. D. & CHERSKII, YU. I., (1978) *Equations of Convolution Type*, Moscow: Nauka (1978) (in Russian)

GANTMACHER, F.R., (1967) *The Theory of Matrices*. 2nd edition, Moscow: Nauka (1967) (in Russian) [English translation (of 1959 edition) reprinted by AMS, Providence, Rhode Island (2000)].

GOHBERG, I.C. & FELDMAN, I.A., (1974) *Convolution Equations and Projection Methods for their Solution,* Transl. Math. Monographs, **41**, Providence, R.I.: AMS.

GOHBERG I., LERER L. & RODMAN L., (1980) *On factorization, indices and completely decomposable matrix polynomials*, Tel-Aviv Univ., *Technical Report* 80-47, pp. 1–72.

GOHBERG, I., KAASHOEK, M.A. & SPITKOVSKY, I.M., (2003) An overview of matrix factorization theory and operator applications, In: *Operator Theory: Advances and Applications,* **141**, 1–102.

GOHBERG, I.TS. & KREIN, M.G., (1958) Systems of integral equations on a half-line with kernels depending on the difference of arguments, *Uspekhi Mat. Nauk,* **XIII** (2 (80)), 3–72 (in Russian).

GOHBERG, I.TS. & KREIN, M.G., (1958) On stable system of partial indices of the Hilbert problem for several unknown functions, *Doklady AN SSSR*, **119** (5), 854–857 (in Russian).

GOHBERG, I. & KRUPNIK, N., (1991-1992) *One-dimensional Linear Singular Integral Equations*, Basel: Birkäuser, **I** (1991), **II** (1992).

GOHBERG, I. & ZUCKER, Y., (1996) On canonical factorization of rational matrix functions, *Integral Equations Operator Theory,* **25** (1), 73–93.

HIGHAM, N.J., (2008) *Functions of Matrices: Theory and Computations,* Philadelphia: SIAM.

HILBERT, D., (1912) *Grundzüge der Integralgleichungen.* Drittes Abschnitt, Leipzig-Berlin.

HORN, R.A. & JOHNSON, C.R., (1985) *Matrix Analysis,* Cambridge: Cambridge University Press.

HORN, R.A. & JOHNSON, C.R., (1991) *Topics in Matrix Analysis,* Cambridge: Cambridge University Press.

JANASHIA, G. & LAGVILAVA, E., (1997) On factorization and partial indices of unitary matrix-functions of one class, *Georgian Math. J.,* **4** (5), 439–442.

JANASHIA, G. & LAGVILAVA, E., (1999) A method of approximate factorization of positive definite matrix functions, *Studia Math.,* **137** (1), 93–100.

JANASHIA, G.A., LAGVILAVA, E.T. & EPHREMIDZE, L.N., (1999) The approximate factorization of positive-definite matrix functions, *Russian Math. Surveys,* **54** (6), 1246–1247; translation from *Uspekhi Mat. Nauk* **54** (6), 161–162.

JANASHIA, G.A., LAGVILAVA, E.T. & EPHREMIDZE, L.N., (2011) A new method of matrix spectral factorization, *IEEE Trans. Inform. Theory,* **57** (4), 2138–2126.

JANASHIA, G.A., LAGVILAVA, E.T. & EPHREMIDZE, L.N., (2013) Matrix spectral factorization and wavelets, *J. Math. Sci.* **195** (4), 445–455.