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Canonical distributions on Riemannian homogeneous k-symmetric spaces

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ABSTRACT

It is known that distributions generated by almost product structures are applicable, in particular, to some problems in the theory of Monge–Ampère equations. In this paper, we characterize canonical distributions defined by canonical almost product structures on Riemannian homogeneous k-symmetric spaces in the sense of types ***AF*** (anti-foliation), ***F*** (foliation), ***TGF*** (totally geodesic foliation). Algebraic criteria for all these types on k-symmetric spaces of orders $k = 4, 5, 6$ were obtained. Note that canonical distributions on homogeneous k-symmetric spaces are closely related to special canonical almost complex structures and f-structures, which were recently applied by I. Khemar to studying elliptic integrable systems.

1. Introduction

Distributions on smooth manifolds can be constructed by various methods. In particular, classical affinor structures such as almost product structures, almost complex structures, f-structures of K. Yano and some others naturally determine two or more (real or complex) complimentary distributions. The distributions generated by almost product structures and their generalizations play a remarkable role in many branches of differential geometry and its applications, specifically, in the problems of contact linearization, contact equivalence, and classification for Monge–Ampère equations (see [1–3]).

Invariant distributions determined by invariant affinor structures on homogeneous manifolds are of especial interest. An important role among homogeneous manifolds of Lie groups is occupied by *homogeneous Φ-spaces* (generalized symmetric spaces), specifically, *homogeneous k-symmetric spaces*, which are generated by Lie group automorphisms Φ of order k ($\Phi^k = id$) [4–7]. The remarkable feature of these spaces is that any homogeneous k-symmetric space $(G/H, \Phi)$ admits a natural associated object, the commutative algebra $\mathcal{A}(\theta)$ [8] of *canonical affinor structures*. This algebra contains well-known classical structures, such as almost complex structures, almost product structures, f-structures ($f^3 + f = 0$), h-structures ($h^3 - h = 0$) (see [8,9]). All classical canonical structures P, J, f, h on homogeneous k-symmetric spaces generate the same distributions called *canonical*. The notable property of the canonical structures and canonical distributions is that all of them are invariant with respect to both the Lie group G and all generalized "symmetries" of order k on G/H.

Up to now, the classical canonical structures above mentioned gave the opportunity to present wealth of invariant examples for Hermitian and *generalized Hermitian geometry* [10] as well as Riemannian geometry (see, e.g., [11,12,9,13]). A new approach was recently created by I. Khemar in [14], where special canonical almost complex structures and f-structures on homogeneous k-symmetric spaces were effectively applied to many geometric constructions in studying elliptic integrable systems.

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We should mention other geometric structures on homogeneous k-symmetric spaces, which are of contemporary interest in geometry and topology. In this respect, in a very recent paper [15] symplectic structures on k-symmetric spaces compatible with the corresponding "symmetries" of order k were studied. In particular, a list of all symplectic 3-symmetric manifolds with simple groups of transvections was given [15]. It is also necessary to notice the direction devoted investigating topology of homogeneous k-symmetric spaces (see [16–18] and others). In particular, it was proved in [17] that all homogeneous k-symmetric spaces of compact simple Lie groups are formal in the sense of Sullivan, and that many of them are not geometrically formal.

In this paper, we study canonical distributions on Riemannian homogeneous k-symmetric spaces $(G/H, g)$, where G is a compact semisimple Lie group and g is any diagonal Riemannian metric on G/H compatible with canonical distributions. We completely characterize so-called base canonical distributions on these spaces in the sense of types ***AF*** (anti-foliation), ***F*** (foliation), ***TGF*** (totally geodesic foliation). In conclusion, the algebraic criteria of including into types ***AF***, ***F***, ***TGF*** for all canonical distributions on Riemannian homogeneous k-symmetric spaces of orders $k = 4, 5, 6$ were obtained.

The paper is organized as follows.

In Section 2, we collect preliminary notions on classical affinor structures on manifolds and distributions of types ***AF***, ***F***, ***TGF*** including some specific details. Besides, we recall the Naveira classification [19] of Riemannian almost product structures.

In Section 3, we study invariant distributions on reductive Riemannian homogeneous spaces generated by invariant almost product structures. Here we prove the criteria under which these distributions belong to types ***AF***, ***F***, ***TGF***.

In Section 4, we give a brief exposition of classical canonical structures on homogeneous regular Φ-spaces, specifically, on homogeneous k-symmetric spaces. Applying the previous results we give a full description of all base canonical distributions on Riemannian homogeneous k-symmetric spaces with diagonal metrics. More exactly, the algebraic criteria for including these distributions into the classes ***AF***, ***F***, ***TGF*** are indicated. In particular, it follows that any base canonical foliation is totally geodesic.

Finally, in Section 5, we examine in detail all canonical distributions (not only base) on Riemannian homogeneous k-symmetric spaces with diagonal metrics for $k = 4, 5, 6$. As a result, a wide collection of new invariant Riemannian foliations as well as the Naveira classes is presented.

2. Riemannian almost product manifolds

2.1. Classical affinor structures and distributions

Let M be a smooth manifold, $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on M. An *affinor structure* F on a manifold is known to be a tensor field of type $(1, 1)$ or, equivalently, a field of endomorphisms acting on its tangent bundle. We recall the following affinor structures of classical types together with their defining conditions:

almost complex structures J $(J^2 = -1)$;

almost product structures P $(P^2 = 1)$;

f-structures $(f^3 + f = 0)$ [20];

f-structures of hyperbolic type or, briefly, *h-structures* $(h^3 - h = 0)$ [10].

Clearly, f-structures and h-structures are generalizations of structures J and P respectively.

Any almost product manifold (M, P) naturally admits two complementary distributions **V** (*vertical*) and **H** (*horizontal*) corresponding to the eigenvalues 1 and -1 of P, respectively.

Let M be an f-manifold, i.e. M is equipped with an f-structure. Then $\mathfrak{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im\, f$ and $\mathcal{M} = Ker\, f$ are distributions, which are usually called the *first* and the *second fundamental distributions* of the f-structure respectively. The number $r = dim\, Im\, f$ is constant at any point of M and called a *rank* of the f-structure. Besides, the number $dim\, Ker\, f = dim\, M - r$ is usually said to be a *deficiency* of the f-structure and denoted by $def\, f$. Obviously, the endomorphisms $l = -f^2$ and $m = 1 + f^2$ are mutually complementary projections on the distributions $\mathcal{L}$ and $\mathcal{M}$ respectively. We note that the restriction F of the f-structure to $\mathcal{L}$ is an almost complex structure, i.e. $F^2 = -1$.

Further, any f-structure on M determines an almost product structure on this manifold in the following way:

$$P = m - l = 2f^2 + 1.$$

It is easy to verify that vertical and horizontal distributions for this structure P are exactly $\mathcal{M}$ and $\mathcal{L}$, respectively. We mention that the converse is not true, i.e. generally an almost product structure does not generate some f-structure on M.

In addition, any h-structure on a manifold M admits three mutually complementary distributions (one of them could be trivial) corresponding to the eigenvalues 1, -1, and 0.

2.2. The Naveira classes of almost product manifolds

Let (M, g) be a Riemannian manifold, P an almost product structure on M. The pair (g, P) is called a Riemannian almost product structure if $g(PX, PY) = g(X, Y)$ for any vector fields X, Y on M.

Riemannian almost product manifold (M, g, P) naturally admits two complementary mutually orthogonal distributions **V** (*vertical*) and **H** (*horizontal*) corresponding to the eigenvalues 1 and -1 of P, respectively. We will denote by A, B, C vertical vector fields, K, L, N horizontal vector fields, and X, Y, Z arbitrary vector fields on M. As usual, denote by ∇ the Levi-Civita connection of the metric g.

In accordance with the Naveira classification [19] there are 36 classes of Riemannian almost product structures (8 types for each of distributions). For the goals of our future consideration, we define here (in terms of vertical distribution) the following types of distributions only:

F (foliation): $\nabla_A(P)B = \nabla_B(P)A$;

AF (anti-foliation): $\nabla_A(P)A = 0$;

TGF (totally geodesic foliation): $\nabla_A P = 0$,

where A and B are vertical vector fields.

Remark 2.1. The defining property ***F*** is equivalent [21] to the condition $[A, B] \in \mathbf{V}$ for any $A, B \in \mathbf{V}$, which is the integrability condition due to the Frobenius Theorem. It is also known [21] that the system of conditions ***AF*** and ***F*** is equivalent to the condition ***TGF***. Besides, it should be mentioned that the condition ***TGF*** is equivalent to the following (formally, weaker) condition $\nabla_A(P)B = 0$ (see [21,19]).

We also note that the class (***TGF***,***TGF***) means a (local) product structure. As to (***TGF***,***AF***), it is a foliation with bundle-like metric, also known as the Reinhart foliation (see [22,23]).

3. Riemannian homogeneous almost product manifolds

Let G be a connected Lie group, H its closed subgroup, $g = \langle\cdot,\cdot\rangle$ an invariant Riemannian structure on the homogeneous space G/H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H respectively. A homogeneous space $(M = G/H, g)$ is said to be *naturally reductive* [24] if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y, Z]_{\mathfrak{m}}\rangle$$

for any $X, Y, Z \in \mathfrak{g}$. Here $\langle\cdot,\cdot\rangle$ denotes the induced metric on $\mathfrak{m}$ and the subscript $\mathfrak{m}$ the projection onto $\mathfrak{m}$ with respect to the reductive decomposition.

Further, suppose P is an invariant Riemannian almost product structure on $(G/H, g)$. The structure P defines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$, where $\mathfrak{m}_+$ ($\mathfrak{m}_-$) is a vertical (horizontal) subspace.

For simplicity, we will denote by the same manner both any invariant structure F on G/H and its value F_o at the point $o = H \in G/H$. It is concerned with all kinds of invariant structures (affinor structures, vector fields, Riemannian metrics, distributions) throughout the rest of the paper.

Now consider invariant almost product structures on Riemannian homogeneous manifolds. Let $(G/H, g = \langle\cdot,\cdot\rangle, P)$ be a naturally reductive homogeneous space. It was proved in [25] that both vertical and horizontal distributions of this structure P are always of type ***AF***. Besides, these distributions may be of type ***F*** (hence, ***TGF***) under simple algebraic criteria. It means that, in accordance with the Naveira classification, there are exactly three classes of invariant naturally reductive almost product structures. They are (***TGF***,***TGF***), (***TGF***,***AF***), (***AF***,***AF***).

It follows that all canonical almost product structures P on Riemannian naturally reductive homogeneous k-symmetric spaces belong to one of these classes. Note that all these classes are realized in this setting. In particular, the situation was characterized for 4-symmetric spaces [25].

We continue this study in more general aspect. Consider a reductive homogeneous space G/H with an invariant almost product structure P and any compatible invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$$

be the corresponding reductive decomposition generated by P. It is well known that the Nomizu function α for the Levi-Civita connection ∇ is of the form

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \tag{1}$$

where $X, Y \in \mathfrak{m}$, and a bilinear symmetric mapping $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined from the equality [24]:

$$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]_{\mathfrak{m}}\rangle + \langle [Z, X]_{\mathfrak{m}}, Y\rangle, \quad \forall Z \in \mathfrak{m}.$$

Theorem 3.1. *Let $(G/H, g = \langle\cdot,\cdot\rangle, P)$ be a Riemannian reductive almost product space. Then the following criteria hold:*

1. *The vertical distribution $\mathfrak{m}_+$ belongs to type **AF** if and only if*

$$U(A, A) \in \mathfrak{m}_+ \quad \textit{for all } A \in \mathfrak{m}_+. \tag{2}$$

*Respectively, the horizontal distribution $\mathfrak{m}_-$ is of type **AF** if and only if*

$$U(K, K) \in \mathfrak{m}_- \quad \textit{for all } K \in \mathfrak{m}_-. \tag{3}$$

2. *The vertical distribution $\mathfrak{m}_+$ belongs to type **F** if and only if*

$$[\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+ \oplus \mathfrak{h}. \tag{4}$$

Respectively, the horizontal distribution $\mathfrak{m}_-$ *is of type* ***F*** *if and only if*

$$[\mathfrak{m}_-, \mathfrak{m}_-] \subset \mathfrak{m}_- \oplus \mathfrak{h}. \tag{5}$$

3. *Finally, the vertical distribution* $\mathfrak{m}_+$ *belongs to type* ***TGF*** *if and only if both conditions (2) and (4) are satisfied. Respectively, the horizontal distribution* $\mathfrak{m}_-$ *belongs to type* ***TGF*** *if and only if both conditions (3) and (5) are satisfied.*

Proof. Obviously, it is sufficient to prove the assertions enumerated above for the vertical distribution only.

1. The defining condition $\nabla_A(P)A = 0$ for the invariant distribution **V** generated by the subspace $\mathfrak{m}_+$ means that

$$\nabla_A PA - P\nabla_A A = \nabla_A A - P\nabla_A A = (1 - P)\nabla_A A = 0$$

for any vertical vector field A on G/H. Now, using the traditional technique of special vector fields in a neighborhood of the point $o = H \in G/H$ (see [26]) it can be shown that the last condition is equivalent to the condition $(1 - P)\alpha(A, A) = 0$ for any $A \in \mathfrak{m}_+$, that is $\alpha(A, A) \in \mathfrak{m}_+$. It follows from (1) that we get the inclusion

$$\frac{1}{2}\,[A, A]_{\mathfrak{m}} + U(A, A) = U(A, A) \in \mathfrak{m}_+ \quad \text{for all } A \in \mathfrak{m}_+.$$

2. On analogy with the previous item we obtain that the condition $\nabla_A(P)B = \nabla_B(P)A$ can be written by means of the Nomizu function α in the following form: $(1 - P)(\alpha(A, B) - \alpha(B, A)) = 0$. Therefore we get

$$\alpha(A, B) - \alpha(B, A) \in \mathfrak{m}_+.$$

Now it follows from (1) that the last condition is equivalent to the condition $[A, B]_{\mathfrak{m}} \in \mathfrak{m}_+$, which means $[\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+ \oplus \mathfrak{h}$.

3. The last assertion follows immediately from Remark 2.1 This concludes the proof.

Remark 3.1. Since the bilinear form U is symmetric the condition (2) can be replaced by the following equivalent condition:

$$U(A, B) \in \mathfrak{m}_+ \quad \text{for all } A, B \in \mathfrak{m}_+.$$

4. Canonical structures and distributions on homogeneous *k*-symmetric spaces

4.1. Homogeneous regular Φ-spaces

We briefly formulate some basic definitions and results related to regular Φ-spaces and canonical affinor structures on them. More detailed information can be found in [8,27,5,7,4,28,9] and others.

Let G be a connected Lie group, Φ its (analytic) automorphism, G^Φ the subgroup of all fixed points of Φ, and G_o^Φ the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition $G_o^\Phi \subset H \subset G^\Phi$. Then G/H is called a *homogeneous Φ-space.*

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, *homogeneous Φ-spaces of order* $k(\Phi^k = id)$ or, in the other terminology, *homogeneous k-symmetric spaces* (see [7]).

For any homogeneous Φ-space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H, \qquad xH \to \Phi(x)H.$$

It is known [4] that S_o is an analytic diffeomorphism of G/H. S_o is usually called a "symmetry" of G/H at the point $o = H$. It is evident that in view of homogeneity the "symmetry" S_p can be defined at any point $p \in G/H$.

Note that there exist homogeneous Φ-spaces that are not reductive. That is why so-called regular Φ-spaces first introduced by N.A. Stepanov [4] are of fundamental importance.

Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\,A$, $\mathfrak{h} \subset \mathfrak{g}_0$. Recall that a homogeneous Φ-space G/H is called a *regular Φ-space* if $\mathfrak{h} = \mathfrak{g}_0$ [4]. Note that other equivalent defining conditions can be found in [8,27].

We formulate two basic facts [4]:

Any homogeneous Φ-space of order k ($\Phi^k = id$) *is a regular Φ-space.*

Any regular Φ-space is reductive. More exactly, the Fitting decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = A\mathfrak{g} \tag{6}$$

is a reductive one.

Decomposition (6) is called the *canonical reductive decomposition* corresponding to a regular Φ-space G/H, and $\mathfrak{m}$ is the *canonical reductive complement*. Besides, this decomposition is obviously φ-invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. We note that θ commutes with any element of the linear isotropy group $Ad(H)$ (see [4]). It also should be noted (see [4]) that $(dS_o)_o = \theta$.

Suppose F is an invariant affinor structure on a homogeneous manifold G/H. Then F is completely determined by its value F_o at the point o, where F_o is invariant with respect to $Ad(H)$. For simplicity, as before, we will denote by the same manner both any invariant structure on G/H and its value at the point o.

Recall [8] that an invariant affinor structure F on a regular Φ-space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ. It follows that any canonical structure is invariant, in addition, with respect to the "symmetries" $\{S_p\}$ of G/H.

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular Φ-space G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Evidently, the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$. As to arbitrary regular Φ-space $(G/H, \Phi)$, the algebraic structure of its commutative algebra $\mathcal{A}(\theta)$ was completely described (see [9]).

4.2. Canonical structures of classical types

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3-symmetric space (see [28,5,29]). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains a rich collection of affinor structures of classical types P, J, f, h.

All the canonical structures of classical type on regular Φ-spaces were completely described [8,9]. In particular, for homogeneous k-symmetric spaces, precise computational formulae were indicated.

We use the notation: $s = [\frac{k-1}{2}]$ (integer part), $u = s$ (for odd k), and $u = s + 1$ (for even k).

Theorem 4.1 ([8,9]). *Let G/H be a homogeneous k-symmetric space.*

1. *All non-trivial canonical f-structures on G/H can be given by the operators*

$$f = \frac{2}{k}\sum_{m=1}^{u}\left(\sum_{j=1}^{u}\zeta_j \sin\frac{2\pi mj}{k}\right)\left(\theta^m - \theta^{k-m}\right),$$

where $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, u$, and not all coefficients ζ_j are zero. In particular, suppose that $-1 \notin$ spec θ. Then the polynomials f define canonical almost complex structures J if and only if all $\zeta_j \in \{-1; 1\}$.

2. *All canonical h-structures on G/H can be given by the polynomials $h = \sum_{m=0}^{k-1} a_m\theta^m$, where:*
   (a) *if* $k = 2n + 1$, *then*
$$a_m = a_{k-m} = \frac{2}{k}\sum_{j=1}^{u}\xi_j \cos\frac{2\pi mj}{k};$$
   (b) *if* $k = 2n$, *then*
$$a_m = a_{k-m} = \frac{1}{k}\left(2\sum_{j=1}^{u}\xi_j \cos\frac{2\pi mj}{k} + (-1)^m\xi_n\right).$$
Here the numbers ξ_j take their values from the set $\{-1; 0; 1\}$ and the polynomials h define canonical structures P if and only if all $\xi_j \in \{-1; 1\}$.

We note that general formulae of this theorem were particularized for homogeneous Φ-spaces of orders 3, 4, 5, 6 in [8,9,13] and some others. In the case $k = 3$ we obtain the well-known canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ (see [28,5,29] and many other papers). As to homogeneous 4-symmetric spaces, they admit, in particular, the only (up to sign) canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ and canonical structure $P = -\theta^2$. Some other canonical structures will be considered in Section 5.

It should be mentioned that canonical structures play an important role in Hermitian and generalized Hermitian geometry. Namely, certain classes of almost Hermitian structures are provided with the remarkable set of invariant examples by means of the canonical almost complex structure on homogeneous 3-symmetric spaces (see, e.g., [5,29]). It turns out that there is also a wealth of invariant examples for the basic classes of metric f-structures. These invariant metric f-structures can be realized on homogeneous k-symmetric spaces with canonical f-structures (see, e.g., [11,9,13]).

We give another explanation for canonical structures P, J, f, h on homogeneous k-symmetric spaces. Let us write the corresponding φ-invariant decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_u, \tag{7}$$

where the subspaces $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ correspond to the spectrum of the operator θ. More precisely, the real subspace $\mathfrak{m}_j$ corresponds to the pair of conjugate kth roots of unity $(\varepsilon^j, \overline{\varepsilon}^j)$, where $\varepsilon = \cos\frac{2\pi}{k} + \sqrt{-1}\sin\frac{2\pi}{k}, j = \overline{1, u}$.

From the procedure of describing the canonical classical structures P, J, f, h (see [8]) it follows that all of them are completely represented in terms of decomposition (7). More exactly, denote by f_i, where $i = 1, 2, \ldots, s$, the *base* canonical f-structure whose image is the subspace $\mathfrak{m}_i$. All the other canonical f-structures are algebraic sums of some distinct base canonical f-structures f_i. In particular, any canonical almost complex structure J is an algebraic sum of s distinct base canonical f-structures f_i. It should be noted that a similar description of canonical almost complex structures on homogeneous k-symmetric spaces was first presented in [5] and used, in particular, for k odd in [7].

On analogy, the *base* canonical h-structure h_i ($i = 1, 2, \ldots, u$) acts on the subspace $\mathfrak{m}_i$ as the identical operator, and its action on the others $\mathfrak{m}_j, j \neq i$ is trivial. All the other canonical h-structures are algebraic sums of some distinct base canonical

h-structures h_i. Specifically, the *base* canonical almost product structure P_i $(i = 1, 2, \ldots, u)$ has $\mathfrak{m}_i$ as a vertical subspace, the others $\mathfrak{m}_j, j \neq i$ form its horizontal subspace. As to the other canonical almost product structures P, they are algebraic sums of u distinct base canonical h-structures h_i.

Furthermore, an invariant distribution D_i on a homogeneous k-symmetric space G/H generated by any subspace $\mathfrak{m}_i$ $(i = 1, 2, \ldots, u)$ is called a *base canonical* distribution. In this setting, a *canonical* distribution on G/H is generated by a direct sum of some subspaces from decomposition (7). It should be stressed that canonical distributions on homogeneous k-symmetric spaces (and even on arbitrary regular Φ-spaces) G/H are invariant with respect to both the Lie group G and all "symmetries" S_p.

4.3. Canonical distributions on k-symmetric spaces

We apply these results for the canonical structures P on homogeneous k-symmetric spaces with the "diagonal" metrics. Let G be a semisimple compact Lie group, B the Killing form of the Lie algebra $\mathfrak{g}$, G/H a homogeneous k-symmetric space. As above, consider the canonical decomposition (7):

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_u,$$

where some subspaces can be trivial. We define the collection of "diagonal" Riemannian metrics $g = \langle \cdot, \cdot \rangle$ on G/H by the formula

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + \cdots + \lambda_u B(X_u, Y_u),$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$ from the above decomposition, $\lambda_i \in \mathbb{R}$, $\lambda_i < 0$. The particular case $\lambda_1 = \cdots = \lambda_u$ gives exactly a naturally reductive Riemannian metric on G/H.

Since diagonal metrics are θ-invariant it follows [9] that any of them is compatible with all canonical classical structures. The Nomizu function α [26] for the Levi-Civita connection ∇ of these "diagonal" metrics was calculated in [30], i.e. the corresponding mapping U is of the form:

$$U(X_i, Y_j)_{\mathfrak{m}_{i\pm j}} = \frac{\lambda_j - \lambda_i}{2\lambda_{i\pm j}} [X_i, Y_j]_{\mathfrak{m}_{i\pm j}}, \qquad U(X_i, Y_i) = U(X_i, Y_j)_{\mathfrak{m}_t} = 0, \tag{8}$$

where $i \geq j$, $\mathfrak{m}_{i+j}$ denotes $\mathfrak{m}_{k-(i+j)}$ if $i + j > u$, λ_{i+j} denotes $\lambda_{k-(i+j)}$ if $i + j > u$, $\mathfrak{m}_t (t \geq 1)$ is any of the subspaces in (7) excluding $\mathfrak{m}_{i-j}$ and $\mathfrak{m}_{i+j}$.

We also recall the important commutator inclusions for the subspaces from the canonical decomposition (7) (see [13]). In the previous notations, they are

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \tag{9}$$

Note that for $k = 2$ this formula gives the well-known classical inclusions for symmetric spaces, namely,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Combining all these results, we can prove the following.

Theorem 4.2. *Any base canonical distribution $\mathfrak{m}_i$, $i = \overline{1, u}$ on Riemannian k-symmetric space $(G/H, g = \langle \cdot, \cdot \rangle)$ is of type **AF** for all "diagonal" metrics g.*

*Further, the distribution $\mathfrak{m}_i$ belongs to **F** (hence, **TGF**) if and only if one of the following cases is realized:*

1. *The subspace $\mathfrak{m}_{2i}$ is trivial.*
2. *The index i satisfies the condition $k = 3i$.*
3. $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.
4. *If $k = 2n$, then $i = n$ (i.e. $\mathfrak{m}_n$ belongs to **F**).*

Proof. From formula (8) it follows that for any subspace $\mathfrak{m}_i$, $i = \overline{1, u}$ we have $U(X_i, Y_i) = 0$, where $X_i, Y_i \in \mathfrak{m}_i$. Now, applying Theorem 3.1(1) we obtain that the distribution $\mathfrak{m}_i$ is of type ***AF***.

To prove the second assertion we use Theorem 3.1(2). In our case the criterion for $\mathfrak{m}_i$ to be of type ***F*** is the following:

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{m}_i \oplus \mathfrak{h}. \tag{10}$$

On the other hand, for the subspace $\mathfrak{m}_i$ from the inclusive relations (9) we have:

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{m}_{2i} + \mathfrak{m}_0 = \mathfrak{m}_{2i} + \mathfrak{h}.$$

Comparing these two inclusions, we conclude that the relation (10) is satisfied in the following cases only:

(1) The subspace $\mathfrak{m}_{2i}$ is trivial. Hence $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.

(2) $\mathfrak{m}_{2i} = \mathfrak{m}_i$. This equality is possible only if $2i > u$. Therefore we have $k - 2i = i$, that is $k = 3i$.

(3) Suppose the subspace $\mathfrak{m}_{2i}$ is non-trivial and $\mathfrak{m}_{2i} \neq \mathfrak{m}_i$. Then $\mathfrak{m}_i$ is of type ***F*** if and only if $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.

(4) Suppose $k = 2n$, then (see Section 4.2) $s = [\frac{2n-1}{2}] = n - 1$. It means that $u = s + 1 = n$. In this case the subspace $\mathfrak{m}_u = \mathfrak{m}_n$ corresponds to the eigenvalue -1 of the operator θ. As a result, we obtain:

$$[\mathfrak{m}_n, \mathfrak{m}_n] \subset \mathfrak{m}_{2n} + \mathfrak{h} = \mathfrak{h} + \mathfrak{h} = \mathfrak{h}.$$

This completes the proof.

Moreover, keeping in mind the result of O. Gil-Medrano (see Remark 2.1) and using Theorem 4.2 we immediately obtain:

Corollary 4.1. *Any base canonical foliation generated by the subspace $\mathfrak{m}_i$ on a Riemannian homogeneous k-symmetric space G/H equipped with arbitrary diagonal metric g is a totally geodesic foliation.*

It should be noted that for base canonical distributions the results of Theorem 4.2 do not depend on the function U. However, for other canonical distributions (e.g., $\mathfrak{m}_i \oplus \mathfrak{m}_j$) the situation is more complicated (see Section 5.3).

5. Canonical distributions on k-symmetric spaces of smaller orders

In conclusion, we completely characterize all canonical distributions on Riemannian k-symmetric spaces of orders 4, 5, and 6. Note that there are no non-trivial (real) canonical distributions on symmetric and 3-symmetric homogeneous spaces.

5.1. Homogeneous 4-symmetric spaces

We recall that any homogeneous 4-symmetric space G/H admits the only (up to sign) canonical almost product structure $P = -\theta^2$ (see [8] or Theorem 3.1 for $k = 4$). The corresponding canonical decomposition (7) is here of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

that is both the canonical distributions $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are base. Further, the commutator relations (9) for these subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the following:

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}.$$

Now, using these relations and Theorem 4.2, we get the following assertions:

Theorem 5.1. *Let G/H be a homogeneous 4-symmetric space, $P = -\theta^2$ the canonical almost product structure, g any diagonal Riemannian metric on G/H, where G is a compact semisimple Lie group. Then the canonical distribution $\mathfrak{m}_2$ is always of type **F** (hence, **TGF**). In addition, $\mathfrak{m}_1$ is of type **F** (hence, **TGF**) if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$. In other words, the structure P belongs to class (**AF**,**TGF**). Moreover, P is of class (**TGF**,**TGF**) if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.*

We notice that Riemannian homogeneous 4-symmetric spaces of the classical compact groups were classified and geometrically described as fiber bundles by J.A. Jimenez (see [31]).

5.2. Homogeneous 5-symmetric spaces

Suppose G/H is a homogeneous 5-symmetric space. Then G/H also admits the only (up to sign) canonical almost product structure (see [32,8]):

$$P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4).$$

In this case, we have again the corresponding canonical decomposition (7) of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the base canonical distributions. As to the commutator relations (9) for these subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$, they are of the following kind:

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1 \oplus \mathfrak{m}_2, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_1 \oplus \mathfrak{h}.$$

Now, applying the same arguments as in Section 4.1, we prove the following.

Theorem 5.2. *Let G/H be a homogeneous 5-symmetric space, where G is a compact semisimple Lie group. Consider the canonical almost product structure P and any diagonal Riemannian metric g on G/H. Then P belongs to class (**AF**,**AF**). Moreover, P is of class (**TGF**,**AF**) if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$. Respectively, P is of class (**AF**,**TGF**) if and only if $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}$. Finally, P belongs to class (**TGF**,**TGF**) if and only if $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$ for $i = 1, 2$.*

5.3. Homogeneous 6-symmetric spaces

Finally, let G/H be a homogeneous 6-symmetric space. Then the corresponding canonical reductive decomposition (7) is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

We also indicate the commutator relations for the subspaces $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$, which follow from (9) for $k = 6$ (see also [33]):

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \qquad [\mathfrak{m}_3, \mathfrak{m}_3] \subset \mathfrak{h},$$
$$[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1 \oplus \mathfrak{m}_3, \qquad [\mathfrak{m}_1, \mathfrak{m}_3] \subset \mathfrak{m}_2, \qquad [\mathfrak{m}_2, \mathfrak{m}_3] \subset \mathfrak{m}_1. \tag{11}$$

Further, all canonical almost product structures on G/H can be represented (up to sign) by the following base canonical structures (see Theorem 4.1):

$$P_1 = \frac{1}{3}(2\theta - \theta^3 + 2\theta^5), \qquad P_2 = \theta^3, \qquad P_3 = \frac{1}{3}(2\theta^2 + 2\theta^4 - 1).$$

Now, applying Theorem 4.2 and commutator relations (11), we get that the base canonical distributions $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$ belong to type ***AF***; moreover, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are of type ***F*** (hence, ***TGF***). In addition, $\mathfrak{m}_1$ is of type ***F*** if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.

In conclusion, we consider canonical distributions of the kind $\mathfrak{m}_i \oplus \mathfrak{m}_j$, where $i, j \in \{1, 2, 3\}$, $i \neq j$. For example, the subspace $\mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{m}_-$ is a horizontal distribution for the canonical structure P_3, that is the subspace $\mathfrak{m}_3 = \mathfrak{m}_+$ is its vertical distribution.

It can be easily shown that the criterion ***AF*** from Theorem 3.1 for the distribution $\mathfrak{m}_- = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is reduced to the inclusion

$$U(X_2, X_1) \in \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $X_1 \in \mathfrak{m}_1$, $X_2 \in \mathfrak{m}_2$. Furthermore, from formula (8) we can obtain:

$$U(X_2, X_1)_{\mathfrak{m}_3} = \frac{\lambda_1 - \lambda_2}{2\lambda_3}[X_2, X_1]_{\mathfrak{m}_3}, \qquad U(X_2, X_1)_{\mathfrak{m}_1} = \frac{\lambda_1 - \lambda_2}{2\lambda_1}[X_2, X_1]_{\mathfrak{m}_1}.$$

Taking into account again commutator relations (11), we conclude that $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ belongs to type ***AF*** if and only if any of the following two conditions is satisfied: (a) $\lambda_1 = \lambda_2$; (b) $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

Now we discuss the condition ***F*** (see Theorem 3.1) for the distribution $\mathfrak{m}_- = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. From the criterion for ***F*** and relations (11) it follows that $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ is of type ***F*** if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

Notice that similar arguments work for the other canonical distributions $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ and $\mathfrak{m}_2 \oplus \mathfrak{m}_3$. Summarizing all these results and using the above notations, we obtain the following.

Theorem 5.3. *Let G/H be a homogeneous 6-symmetric space, where G is a compact semisimple Lie group. Suppose g is any diagonal Riemannian metric on G/H represented by the collection $(\lambda_1, \lambda_2, \lambda_3)$. Then:*

1. $\mathfrak{m}_2$ *and* $\mathfrak{m}_3$ *are of type **TGF**.*
2. $\mathfrak{m}_1$ *belongs to type **TGF** if and only if* $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.
3. $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *is of type **AF** if and only if any of the following two conditions is satisfied: (a)* $\lambda_1 = \lambda_2$*; (b)* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.
4. $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *is of type **F** if and only if* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$*. This is also a criterion for type **TGF**.*
5. $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type **AF** if and only if any of the following two conditions is satisfied: (a)* $\lambda_1 = \lambda_3$*; (b)* $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$.
6. $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ *is of type **F** if and only if both the following relations hold:* $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$, $[\mathfrak{m}_1, \mathfrak{m}_3] = 0$*. This is also a criterion for type **TGF**.*
7. $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type **AF** if and only if any of the following two conditions is satisfied: (a)* $\lambda_2 = \lambda_3$*; (b)* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$.
8. $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ *is of type **F** if and only if* $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$*. This is also a criterion for type **TGF**.*

This theorem gives the opportunity to characterize the Naveira classes for all combinations of the above canonical distributions. As an example, the canonical structure P_3 belongs to the class (***TGF***,***TGF***) if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$.

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