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# On Banach–Mazur limits

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**Abstract**

A positive functional $x^*$ on the space $\ell_\infty$ of all bounded sequences is called a *Banach–Mazur limit* if $\|x^*\| = 1$ and $x^*x = x^*Tx$ for all $x = (x_1, x_2, \ldots) \in \ell_\infty$, where $T$ is the forward shift operator on $\ell_\infty$, i.e., $Tx = (0, x_1, x_2, \ldots)$. The set of all Banach–Mazur limits is denoted by BM and a collection of extreme points of BM is denoted by ext BM. Let

$$ac_0 = \{x \in \ell_\infty : x^*x = 0 \text{ for all } x^* \in \mathrm{BM}\}.$$

The following sequence spaces

$$\mathcal{D}(ac_0) = \{x \in \ell_\infty : x \cdot ac_0 \subseteq ac_0\} \quad \text{and} \quad \mathcal{I}(ac_0) = ac_0^+ - ac_0^+$$

are studied. In particular, if $z \in \ell_\infty$ then $z \in \mathcal{D}(ac_0)$ iff $z - Tz \in \mathcal{I}(ac_0)$; moreover, $z \in \mathcal{D}(ac_0)$ iff $x^*\{n : |z_n - x^*z| \geq \epsilon\} = 0$ for all $\epsilon > 0$ and $x^* \in \text{ext BM}$. Order properties of Banach–Mazur limits are considered. Some properties of ext BM are derived. We used the representation of functionals $x^* \in \mathrm{BM}$ as Borel measures on $\beta\mathbb{N} \setminus \mathbb{N}$. The cardinalities of some subset of BM are given. We also consider some questions of the probability theory for finite additive measures. E.g., for every $x^* \in \mathrm{BM}$ there exists an element $x \in \ell_\infty$ such that the distribution function $F_{x^*,x}(t) = x^*\{n : x_n \leq t\}$ is continuous on $\mathbb{R}$. Two definitions of a variance are suggested. It is shown that Radon–Nikodym theorem is not valid for finite additive measures: the relations $0 \leq x^* \leq y^* \in \ell_\infty^*$ do not imply the existence of $w \in \ell_\infty$ satisfying $x^*x = y^*(wx)$ for all $x \in \ell_\infty$.


*Keywords:* Banach–Mazur limit; Stone-Čech compactification $\beta\mathbb{N}$; Finite additive probability measure

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## 1. Introduction and preliminaries

Let $\ell_\infty$ be the linear space of all (real) bounded sequences $x = (x_1, x_2, \ldots)$ under the natural algebraic operations. Under the sup norm $\|x\| = \sup_n |x_n|$, the space $\ell_\infty$ is a Banach space. As usual, for two sequences $x, y \in \ell_\infty$, we write $x \geq y$ (or $y \leq x$) if $x_n \geq y_n$ for all $n$. Under this ordering, $\ell_\infty$ is a Riesz space and, moreover, is even a Banach lattice with positive cone $\ell_\infty^+ = \{x \in \ell_\infty : x \geq 0\}$. Furthermore, for any $x, y \in \ell_\infty$ the multiplication can be defined by $xy = (x_1y_1, x_2y_2, \ldots)$. Therefore, $\ell_\infty$ is a commutative Banach algebra with unit $\mathbf{e} = (1, 1, \ldots)$.

As usual, the norm dual space $\ell_\infty^*$ of $\ell_\infty$ is a collection of all (linear, bounded) functionals on $\ell_\infty$. The space $\ell_\infty^*$ is a Banach lattice. It is well known (see [2, p. 539]) that the band $(\ell_\infty^*)_n^\sim$ of all order continuous functionals on $\ell_\infty$ is lattice isometric onto the space $\ell_1$ of all absolutely summable sequences. Therefore, the decomposition

$$\ell_\infty^* = \ell_1 \oplus \ell_1^{\mathrm{d}} = \ell_1 \oplus \{x^* \in \ell_\infty^* : x^*(c_0) = \{0\}\}$$

holds, where $c_0$ is the space of all sequences converging to zero (for another representation of $\ell_\infty^*$, see (20) and (28)).

A linear functional $x^*$ on $\ell_\infty$ is called a *Banach–Mazur limit* (see, e.g., [2, Section 16.10]) if

(a) $x^*$ is a positive functional, i.e., $x^*x \geq 0$ for each $x \in \ell_\infty^+$;
(b) $x^*(x_1, x_2, x_3, \ldots) = x^*(0, x_1, x_2, \ldots)$ for each $x \in \ell_\infty$;
(c) $x^*\mathbf{e} = 1$.

The set of all Banach–Mazur limits is denoted by BM. Other names used for Banach–Mazur limits are *Banach limits* and *generalized limits*. In 1929, Mazur has proved that BM $\neq \emptyset$. Since then Banach–Mazur limits have been investigated in various ways by many authors (see, e.g., [1,4,5,8–13,15,16] and the references in them). Our paper extends this line of research.

We recall some results about Banach–Mazur limits which will be used later on. As follows at once from condition (a) above, every functional $x^* \in$ BM is bounded, i.e., $x^* \in \ell_\infty^*$. Condition (c), in it turn, implies the equality $\|x^*\|_{\ell_\infty^*} = 1$, i.e., $x^*$ belongs to the positive part of the unit sphere $S_{\ell_\infty^*}^+$ of $\ell_\infty^*$. As is easy to see, the set BM is convex and $\sigma(\ell_\infty^*, \ell_\infty)$-compact and, hence, by Krein–Milman theorem [3, p. 137], it is the $\sigma(\ell_\infty^*, \ell_\infty)$-closed convex hull of the set ext BM of its extreme points. On the other hand, the set ext BM is not $\sigma(\ell_\infty^*, \ell_\infty)$-closed [11]. For $x \in \ell_\infty$, we put $\tau(x) = \max_{x^* \in \mathrm{BM}} x^*x = \max_{x^* \in \mathrm{ext\ BM}} x^*x$. We have the next identities [15]

$$\tau(x) = \lim_{n\to\infty} \sup_m \frac{1}{n} \sum_{i=0}^{n-1} x_{m+i} \tag{1}$$

and

$$\min_{x^* \in \mathrm{BM}} x^*x = \min_{x^* \in \mathrm{ext\ BM}} x^*x = \lim_{n\to\infty} \inf_m \frac{1}{n} \sum_{i=0}^{n-1} x_{m+i}.$$

Next, as is easy to see, if $x \in c_0$ then $x^*x = 0$ for all $x^* \in$ BM. A sequence $x \in \ell_\infty$ is said to be *almost converging to zero* [9] whenever $x^*x = 0$ for all $x^* \in$ BM. G. Lorentz proved in [9] that a sequence $x \in \ell_\infty$ almost converging to zero iff

$$\lim_{n\to\infty} \frac{x_m + \cdots + x_{m+n-1}}{n} = 0$$

uniformly in $m$. The collection of all sequences almost converging to zero is denoted by $ac_0$. Obviously, $ac_0$ is a closed subspace of $\ell_\infty$ and the inclusion $c_0 \subseteq ac_0$ holds. This inclusion is proper. Indeed, if $\{n_k\}$ is a subsequence of $\mathbb{N}$ and $n_{k+1} - n_k \to \infty$ as $k \to \infty$ then [1] the characteristic sequence $\chi_{\{n_1,n_2,\dots\}} \in ac_0$.

If $T$ is the forward shift operator on $\ell_\infty$, i.e., $Tx = (0, x_1, x_2, \dots)$, then condition (b) of the definition of a Banach–Mazur limit is equivalent to $x^* = T^*x^*$, where $T^*$ is the adjoint operator of $T$. In particular, the space $ac_0$ is $T$-invariant and the inclusion $\mathrm{BM} \subseteq N(I - T^*)$ holds, where $N(I - T^*)$ is the null space of $I - T^*$. Moreover, the set BM is the positive part of the unit sphere of the $AL$-space $N(I - T^*)$, whence, for $z^* \in \mathrm{BM}$, we have $z^* \in \operatorname{ext} \mathrm{BM}$ iff $z^*$ is an atom in $N(I - T^*)$ (see Section 4 for the detailed discussion). If $U$ is the backward shift operator on $\ell_\infty$, i.e., $Ux = (x_2, x_3, x_4, \dots)$, then for the ranges of the operators $I - T$ and $I - U$, we have $R(I - T) = R(I - U) = bs$, where $bs$ is the space of bounded series defined by

$$bs = \left\{x \in \ell_\infty : \sup_n \left|\sum_{i=1}^{n} x_i\right| < \infty\right\},$$

and, hence $N(I - T^*) = N(I - U^*)$. The relation $\overline{bs} = ac_0$ holds (see, e.g., [1]), where the closure was taken in the norm topology of $\ell_\infty$.

The paper is organized as follows. In Section 2, the stabilizer $\mathcal{D}(ac_0)$ of the space $ac_0$ is studied. In Section 3, Banach–Mazur limits are considered as measures on the Stone–Čech compactification $\beta\mathbb{N}$. Section 4 discusses, in particular, the cardinalities of some subsets of BM. In the last section, the results obtained in the preceding ones are considered from the viewpoint of the probability theory. It allows, on the one hand, in a new fashion to look at some properties of Banach–Mazur limits and, on the other hand, to take a step in the study of finite additive probability measures.

For any unexplained terminology, notions, and elementary properties on ordered linear spaces, we refer to [2,3]. For information on the Stone–Čech compactification, we suggest [7]. More details on weak topologies on Banach spaces can be found in [3, Chapter 3] (see also [6, Chapter 5] and [2, Chapters 5–7]). We refer the reader to [14] for necessary information from the probability theory. In the sequel, unless stated otherwise, considering some topology, we will assume the norm topology of a given normed space. Furthermore, the case of a sequence $x$ as an element of the space $\ell_\infty$ (or, more generally, of the space $s$ of all real sequences) and the case of a sequence of elements $\{x_n\}$ in some set $A$ of an arbitrary nature should differ. The *support* of an arbitrary *function* $f : A \to \mathbb{R}$ is the set $\operatorname{supp} f = \{a \in A : f(a) \neq 0\}$. Next, we put $\mathbf{e}_n = \chi_{\{n\}}$, where $n \in \mathbb{N}$. For a subset $B$ of $\mathbb{N}$ the operator $P_B$ on $\ell_\infty$ is defined by $P_B x = \chi_B x$. For an arbitrary (linear) subspace $X$ of $s$ and a subset $B$ of $\mathbb{N}$ the expression $B \in X$ ($B \notin X$) means the validity of the relation $\chi_B \in X$ ($\chi_B \notin X$); if $x^*$ is a functional on $X$ and $B \in X$ then we put $x^*B = x^*\chi_B$.

## 2. The sequence space $\mathcal{D}(ac_0)$

The *stabilizer* of the space $ac_0$ is called the set

$$\mathcal{D}(ac_0) = \{z \in s : z \cdot ac_0 \subseteq ac_0\}.$$

The stabilizer was introduced in [10]; see also [1]. This section is a continuation of research which was begun in these two papers.

For an arbitrary functional $x^* \in \mathrm{BM}$, we put

$$\mathcal{D}_{x^*} = \{z \in s : z \cdot ac_0 \subseteq N(x^*)\}.$$

where $N(x^*)$ is the null space of the functional $x^*$.

Obviously, $\mathbf{e} \in \mathcal{D}_{x^*}$. The inclusion $\mathcal{D}_{x^*} \subseteq \ell_\infty$ holds. Indeed, if $z \in \mathcal{D}_{x^*} \setminus \ell_\infty$ then for some subsequence $\{n_k\}$ of indexes, the relation $z_{n_k} \to \infty$ is valid. We can assume that the set $\{n_1, n_2, \ldots\} \in ac_0$. Thus, the sequence $z \cdot \chi_{\{n_1,n_2,\ldots\}} \in N(x^*)$ and, in particular, is bounded, a contradiction. Next, the subspace $\mathcal{D}_{x^*}$ is $T$-invariant. To see this, for arbitrary sequences $z \in \mathcal{D}_{x^*}$ and $y \in ac_0$, we have

$$x^*(Tz \cdot y) = x^*(U(Tz \cdot y)) = x^*(z \cdot Uy) = 0,$$

as required. Using the identities

$$ac_0 = \bigcap_{x^* \in \mathrm{BM}} N(x^*) = \bigcap_{x^* \in \mathrm{ext\ BM}} N(x^*),$$

we obtain

$$\mathcal{D}(ac_0) = \bigcap_{x^* \in \mathrm{BM}} \mathcal{D}_{x^*} = \bigcap_{x^* \in \mathrm{ext\ BM}} \mathcal{D}_{x^*} \tag{2}$$

and, hence, $\mathcal{D}(ac_0)$ is a closed $T$-invariant subspace of $\ell_\infty$.

We need the next variant of the classical Chebyshev's inequality.

**Lemma 1.** *For $x \in \ell_\infty,\ x^* \in \ell_\infty^*$, and a number $\lambda > 0$, we have the following inequalities*

$$|x^*|\{n : |x_n| \geq \lambda\} \leq \lambda^{-1}|x^*|\,|x| \tag{3}$$

*and*

$$|x^*|\{n : |x_n - x^*x| \geq \lambda\} \leq \min\{\lambda^{-1}|x^*|\,|x - (x^*x)\mathbf{e}|,\ \lambda^{-2}|x^*|((x - (x^*x)\mathbf{e})^2)\}.$$

**Proof.** The first inequality follows at once from the next relations

$$|x^*|\,|x| \geq |x^*|P_{\{n:|x_n|\geq\lambda\}}|x| \geq \lambda|x^*|\{n : |x_n| \geq \lambda\}$$

and the second inequality is a simple consequence of the former. □

**Corollary 2.** *If $x \in \ell_\infty, x^* \in \ell_\infty^*$, and $|x^*|\,|x - (x^*x)\mathbf{e}| = 0$ then for every $\epsilon > 0$ the equality $|x^*|\{n : |x_n - x^*x| \geq \epsilon\} = 0$ holds.*

**Theorem 3.** *For an element $z \in \ell_\infty$ and a functional $x^* \in \mathrm{ext\ BM}$ the following statements are equivalent:*

(a) $z \in \mathcal{D}_{x^*}$;

(b) $z \cdot N(x^*) \subseteq N(x^*)$;

(c) *For every $x \in \ell_\infty$, we have*

$$x^*(xz) = (x^*x)(x^*z); \tag{4}$$

(d) $x^*|z - (x^*z)\mathbf{e}| = 0$;

(e) *For every $y \in \mathcal{D}_{x^*}$, we have* $x^*\Big|\,|y - z| - (x^*|y - z|)\mathbf{e}\Big| = 0$;

(f) *For every $y \in \mathcal{D}_{x^*}$, we have* $x^*\Big|(x^*|z|)|y - z| - (x^*|y - z|)|z|\Big| = 0$;

(g) $x^*|z - Tz| = 0$;

(h) *For every $\epsilon > 0$, we have*

$$x^*\{n : |z_n - x^*z| \geq \epsilon\} = 0.$$

**Proof.** The implication (b) $\Longrightarrow$ (a) is obvious.

(a) $\Longrightarrow$ (c) If $x^*|z| = 0$ then the assertion is clear. Let $x^*|z| > 0$. Define the functional $x_z^*$ on $\ell_\infty$ via the formula

$$x_z^* x = x^*(xz). \tag{5}$$

The relations [1] $|x_z^*| \le \|z\| x^*$ and $\|x_z^*\|_{\ell_\infty^*} = x^*|z|$ are valid. Since $z \in \mathcal{D}_{x^*}$, we obtain $x_z^*(ac_0) = \{0\}$ and so $x_z^* \in N(I - T^*)$, whence $\frac{|x_z^*|}{x^*|z|} \in \mathrm{BM}$. Now, taking into account the inclusion $x^* \in \mathrm{ext\,BM}$, we infer $|x_z^*| = (x^*|z|)x^*$. Choose a scalar $\alpha$ satisfying $z + \alpha\mathbf{e} \ge 0$. Since $z + \alpha\mathbf{e} \in \mathcal{D}_{x^*}$, the last equality implies $x^*(x(z + \alpha\mathbf{e})) = x^*x \cdot x^*(z + \alpha\mathbf{e})$ and so $x^*(xz) = (x^*x)(x^*z)$ for all $x \in \ell_\infty$.

(c) $\Longrightarrow$ (b) For every $x \in N(x^*)$, we have $x^*(xz) = x^*x \cdot x^*z = 0$, i.e., $xz \in N(x^*)$.

Thus, the equivalence of the first three statements has been established. It follows from (b) that $\mathcal{D}_{x^*}$ is a *closed subalgebra* of $\ell_\infty$. Consequently, as every closed subalgebra of the space $\ell_\infty$ containing the unit $\mathbf{e}$ or, in general, of the space $C(K)$ of all continuous functions on some (Hausdorff) compact space $K$, $\mathcal{D}_{x^*}$ is a *Riesz subspace* of $\ell_\infty$.

Now we verify that the equivalent statements (a)–(c) imply the validity of each of the statements (d)–(g). To this end, if an element $z \in \mathcal{D}_{x^*}$ then, in view of (c), for an arbitrary element $x \in \ell_\infty$ the relation

$$x^*(x(z - (x^*z)\mathbf{e})) = 0 \tag{6}$$

holds. From this, (d) follows at once. Therefore, if $y \in \mathcal{D}_{x^*}$ then, since $|y - z| \in \mathcal{D}_{x^*}$, we have $x^*\Big|\,|y - z| - (x^*|y - z|)\mathbf{e}\Big| = 0$ and (e) has been proved. On the other hand, using (c) once more, we obtain $x^*(x|z|) \cdot x^*|y - z| = x^*|z| \cdot x^*(x|y - z|)$ for every element $x \in \ell_\infty$. Thus, $x^*\Big|(x^*|z|)|y - z| - (x^*|y - z|)|z|\Big| = 0$ and (f) is proven. Next, since $Tz \in \mathcal{D}_{x^*}$, we have

$$x^*(x(z - Tz)) = x^*(xz) - x^*(x \cdot Tz) = x^*x \cdot x^*z - x^*x \cdot x^*(Tz) = 0$$

and (g) has been established.

(d) $\Longrightarrow$ (b) In view of (6), for $y \in N(x^*)$, we have $x^*(yz) = 0$, i.e., $z \cdot N(x^*) \subseteq N(x^*)$.

(e) $\Longrightarrow$ (a) Using the preceding implication, we obtain $|y - z| \in \mathcal{D}_{x^*}$ for $y \in \mathcal{D}_{x^*}$. In particular, for $y = -\|z\|\mathbf{e}$ the relations $z + \|z\|\mathbf{e} = |-\|z\|\mathbf{e} - z| \in \mathcal{D}_{x^*}$ hold, whence $z \in \mathcal{D}_{x^*}$.

(f) $\Longrightarrow$ (e) If $x^*|z| = 0$ then $z \in \mathcal{D}_{x^*}$ and, as showed above, (e) is valid. Let $x^*|z| > 0$. For every $\beta \ge \|z\|$ the equality

$$x^*\Big|(x^*|z|)(\beta\mathbf{e} - z) - (x^*(\beta\mathbf{e} - z))|z|\Big| = 0$$

holds and, consequently, $x^*\Big|(x^*|z|)\mathbf{e} - \frac{x^*|z|}{\beta}z - |z| + \frac{x^*z}{\beta}|z|\Big| = 0$. Letting $\beta \to +\infty$, we have $x^*\Big|\,|z| - (x^*|z|)\mathbf{e}\Big| = 0$. Whence, using our condition once more and the last identity, for every $y \in \mathcal{D}_{x^*}$, we obtain

$$\begin{aligned}
x^*\Big|\,|y - z| - (x^*|y - z|)\mathbf{e}\Big| &= \frac{1}{x^*|z|}x^*\Big|(x^*|z|)|y - z| - (x^*|y - z| \cdot x^*|z|)\mathbf{e}\Big| \\
&\le \frac{1}{x^*|z|}x^*\Big|(x^*|y - z|)|z| - (x^*|y - z| \cdot x^*|z|)\mathbf{e}\Big| \\
&= \frac{x^*|y - z|}{x^*|z|}x^*\Big|\,|z| - (x^*|z|)\mathbf{e}\Big| = 0,
\end{aligned}$$

as desired.

(g) $\Longrightarrow$ (a) For an arbitrary element $x \in \ell_\infty$, we have the equalities

$$\begin{aligned}0 &= x^*(x(z - Tz)) = x^*(xz) - x^*(U(x \cdot Tz)) \\ &= x^*(xz) - x^*((Ux)z) = x^*(((I - U)x)z),\end{aligned}$$

i.e., $z \cdot bs \subseteq N(x^*)$ and, hence, $z \cdot ac_0 \subseteq N(x^*)$. Finally, $z \in \mathcal{D}_{x^*}$.

The implication (d) $\Longrightarrow$ (h) follows at once from Corollary 2.

(h) $\Longrightarrow$ (b) For an arbitrary element $x \in N(x^*)$ and a number $\epsilon > 0$, we have

$$\begin{aligned}x^*(xz) &= x^*(x(z - (x^*z)\mathbf{e})) \\ &= x^*(P_{\{n:|z_n - x^*z| \geq \epsilon\}}x(z - (x^*z)\mathbf{e})) + x^*(P_{\{n:|z_n - x^*z| < \epsilon\}}x(z - (x^*z)\mathbf{e})) \leq \epsilon x^*|x|.\end{aligned}$$

Letting $\epsilon \downarrow 0$, we obtain $xz \in N(x^*)$. □

We note that the identity $x^*\Big|\,|\mathbf{e} - z| - (x^*|\mathbf{e} - z|)\mathbf{e}\Big| = 0$ with $z \in \ell_\infty$ does not imply the inclusion $z \in \mathcal{D}_{x^*}$. To see this, it suffices to consider an arbitrary element $z$ such that, on the one hand, $|\mathbf{e} - z| = \mathbf{e}$ and, on the other hand, $z \notin N(x^*)$ (e.g., $z = (0, 2, 0, 2, \ldots)$).

**Corollary 4.** *The restriction of* $x^* \in \text{ext BM}$ *on* $\mathcal{D}_{x^*}$ *is a lattice and algebraic homomorphism.*

**Proof.** As was mentioned in the preceding theorem, the space $\mathcal{D}_{x^*}$ is a closed subalgebra and a Riesz subspace of $\ell_\infty$. According to part (c) of this theorem, $x^*$ is multiplicative on $\mathcal{D}_{x^*}$. Next, consider an element $z \in \mathcal{D}_{x^*}$ and find a sequence $x \in \ell_\infty$ satisfying $xz = |z|$ and $|x| = \mathbf{e}$. Using (4), we get $x^*|z| = |x^*x|\,|x^*z| \leq |x^*z| \leq x^*|z|$. Consequently, $x^*$ is a lattice homomorphism. As a matter of fact, the next result holds: *If $J$ is a closed subalgebra of $\ell_\infty$ (or, in general, of the space $C(K)$) and the unit* $\mathbf{e} \in J$ *then a functional $x^* \in J^*$ with* $x^*\mathbf{e} = 1$ *is a lattice homomorphism iff $x^*$ is an algebraic homomorphism.* □

**Corollary 5.** *For* $x^* \in \text{ext BM}$ *and a subset $D$ of $\mathbb{N}$ the equality $x^*(D \bigtriangleup (D + 1)) = 0$ implies $x^*D \in \{0, 1\}$.*

**Proof.** We recall first that the symmetric difference $A \bigtriangleup B$ of sets $A$ and $B$ defined via the formula $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$. The relations $0 = x^*(D \bigtriangleup (D + 1)) = x^*|\chi_D - T\chi_D|$ hold. In view of Theorem 3(g), $D \in \mathcal{D}_{x^*}$. Whence, taking into account the preceding corollary, we infer $x^*D = (x^*D)^2$ and so $x^*D \in \{0, 1\}$. □

Before proceeding further, we recall that in the space $\ell_\infty$ the notions of algebraic ideal and of order ideal coincide (see, e.g., [1]). Thus, in the sequel, we shall simply use the term *ideal*. On the other hand, the space $ac_0$ is not an ideal in $\ell_\infty$. There exists (see [1]) the maximal (by inclusion) ideal in $ac_0$ which is called an *ideal stabilizer* of $ac_0$ and will be denoted by $\mathcal{I}(ac_0)$. For a sequence $z \in \ell_\infty$ the inclusion $z \in \mathcal{I}(ac_0)$ holds iff

$$\lim_{n\to\infty} \frac{|z_m| + \cdots + |z_{m+n-1}|}{n} = 0 \tag{7}$$

uniformly in $m$. Moreover, the equality

$$\mathcal{I}(ac_0) = \{x - y : x, y \in ac_0 \text{ and } x, y \geq 0\}$$

is valid and, in particular, the ideal $\mathcal{I}(ac_0)$ is $T$-invariant. Obviously, $\mathcal{I}(ac_0)$ is a closed ideal and the relations $c_0 \subsetneq \mathcal{I}(ac_0) \subsetneq \mathcal{D}(ac_0)$ hold.

**Theorem 3′.** *For an element $z \in \ell_\infty$ the following statements are equivalent:*

(a) $z \in \mathcal{D}(ac_0)$*;*
(b) *For every* $x^* \in \text{ext BM}$*, we have* $z \cdot N(x^*) \subseteq N(x^*)$*;*
(c) *For every* $x \in \ell_\infty$ *and* $x^* \in \text{ext BM}$*, we have*

$$x^*(xz) = (x^*x)(x^*z); \tag{8}$$

(d) *For every* $x^* \in \text{ext BM}$*, we have* $x^*|z - (x^*z)\mathbf{e}| = 0$*;*
(e) *For every* $y \in \mathcal{D}(ac_0)$ *and* $x^* \in \text{ext BM}$*, we have* $x^*\Big| |y - z| - (x^*|y - z|)\mathbf{e}\Big| = 0$*;*
(f) *For every* $y \in \mathcal{D}(ac_0)$ *and* $x^* \in \text{ext BM}$*, we have* $x^*\Big|(x^*|z|)|y - z| - (x^*|y - z|)|z|\Big| = 0$*;*
(g) $z - Tz \in \mathcal{I}(ac_0)$*;*
(h) *For every* $\epsilon > 0$ *and* $x^* \in \text{ext BM}$*, we have*

$$x^*\{n : |z_n - x^*z| \geq \epsilon\} = 0 \tag{9}$$

*(i.e., the sequence $z$ converges "scatterly" with respect of $x^*$ to $x^*z$).*

**Proof.** It follows immediately from the preceding theorem and the equalities (2) that the statement (a) implies each of the statements (b)–(h) and each of the statements (b)–(d), (g), and (h) implies (a). The implications (f) $\Longrightarrow$ (e) $\Longrightarrow$ (a) can be checked as analogous implications of Theorem 3. □

The equivalence of parts (a), (c), and (d) of the preceding theorem was earlier established in [1] (see also [10]). As follows from part (c), *every functional* $x^* \in \text{BM}$ *which belongs to* $\overline{\text{ext BM}}^{\sigma(\ell_\infty^*, \ell_\infty)}$ *is multiplicative on* $\mathcal{D}(ac_0)$. It is not known if the converse holds.

Now we will discuss the question about the validity of the inclusion $D \in \mathcal{D}(ac_0)$ for some subset $D$ of $\mathbb{N}$. Obviously, if a subset $D$ is finite then $D, \mathbb{N} \setminus D \in \mathcal{D}(ac_0)$. The next result was obtained in [13].

**Corollary 6.** *Suppose that $D$ is a subset of $\mathbb{N}$ defined by $D = \bigcup_{k=1}^{\infty}[d_{2k-1}, d_{2k} - 1]$ with $d_k \in \mathbb{N}$ and $d_k < d_{k+1}$ for all $k \in \mathbb{N}$. Then $D \in \mathcal{D}(ac_0)$ iff*

$$\lim_{j\to\infty} \frac{d_{k+j-1} - d_k}{j} = \infty \tag{10}$$

*uniformly in $k$.*

**Proof.** In view of part (g) of the preceding theorem, we have that the set $D \in \mathcal{D}(ac_0)$ iff the set $B = \{d_1, d_2, \ldots\} \in \mathcal{I}(ac_0)$. The latter implies $\lim_{j\to\infty} \frac{j-1}{d_{k+j-1} - d_k} = 0$ uniformly in $k$ and the necessity follows. For the converse, let (10) hold. Put $d_0 = 0$. For arbitrary numbers $m, n \in \mathbb{N}$, we define indexes $k_m \in \mathbb{N}$ and $j_{m,n} \in \{-1, 0\} \cup \mathbb{N}$ by the following manner. Find an index $k_m$ satisfying $d_{k_m - 1} < m \leq d_{k_m}$. Next, if $m + n - 1 < d_{k_m}$ (e.g., $n = 1$ and $m < d_{k_m}$) then we put $j_{m,n} = -1$. If $m + n - 1 \geq d_{k_m}$ then we pick $j_{m,n}$ such that $d_{k_m + j_{m,n}} \leq m + n - 1 < d_{k_m + j_{m,n} + 1}$. Evidently,

$$n = m + n - 1 - m + 1 \geq d_{k_m + j_{m,n}} - d_{k_m} + 1 \geq d_{k_m + j_{m,n}} - d_{k_m}. \tag{11}$$

On the other hand,

$$\frac{1}{n}\sum_{i=m}^{m+n-1} (\chi_B)_i = \frac{\text{card}\,(B \cap [m, m + n - 1])}{n} = \frac{j_{m,n} + 1}{n}.$$

Consequently, it suffices to check the equality $\lim_{n\to\infty}\frac{j_{m,n}+1}{n}=0$ uniformly in $m$, which, in view of (7), implies the inclusion $B\in\mathcal{I}(ac_0)$. To this end, proceeding by contradiction, we find $\epsilon>0$ and subsequences $\{m_r\}$ and $\{n_r\}$ of $\mathbb{N}$ satisfying the relations $\frac{j_{m_r,n_r}+1}{n_r}>\epsilon$ for all $r$ and $n_r\to\infty$ as $r\to\infty$, whence $j_{m_r,n_r}\to\infty$. Taking into account the relations (11), we have $\epsilon<\frac{j_{m_r,n_r}+1}{d_{k_{m_r}+j_{m_r,n_r}}-d_{k_{m_r}}}$. The latter contradicts (10). $\square$

**Corollary 7.** *Suppose that $\{d'_k\}$ and $\{d''_k\}$ are two sequences in $\mathbb{N}$ satisfying $d'_k\le d''_k$ and $\lim_{k\to\infty}(d''_k-d'_k)=\infty$. Then the next statements hold:*

(a) *The subset $D=\bigcup_{k=1}^{\infty}[d'_k,d''_k]$ of $\mathbb{N}$ belongs to $\mathcal{D}(ac_0)$;*
(b) *If $x\in\ell_\infty$ and $d''_k\le d'_{k+1}$ for sufficiently large $k$ then the element $z=\sum_{k=1}^{\infty}x_k\chi_{[d'_k,d''_k]}$ belongs to $\mathcal{D}(ac_0)$.*

**Proof.** (a) As is easy to see, we can assume that the sets $[d'_k,d''_k]$ are pairwise disjoint and $d''_k+1<d'_{k+1}$ for all $k$. Consequently, $\lim_{k\to\infty}d'_k=\lim_{k\to\infty}d''_k=\infty$ and, hence, both sets $\{d'_1,d'_2,\dots\},\{d''_1,d''_2,\dots\}\in\mathcal{I}(ac_0)$. Thus,

$$\chi_D-T\chi_D=\chi_{\{d'_1,d'_2,\dots\}}-\chi_{\{d''_1+1,d''_2+1,\dots\}}\in\mathcal{I}(ac_0).$$

In view of Theorem 3′(g), $D\in\mathcal{D}(ac_0)$.

The statement (b) can be proved in a similar manner. $\square$

It should be noted that the relation (9) does not hold for an arbitrary functional $x^*\in$ BM and $z\in\mathcal{D}(ac_0)$. In fact, consider the sequence $z=(1,0,1,1,0,0,1,1,1,\dots)$. According to the preceding corollary, $z\in\mathcal{D}(ac_0)$. Fix a number $\lambda\in(0,1)$. As follows easily from (1), there exists a functional $x^*_\lambda\in$ BM satisfying $x^*_\lambda z=\lambda$. Next, for every $\epsilon\in(0,\min\{\lambda,1-\lambda\})$, we have $x^*_\lambda\{n:|z_n-\lambda|\ge\epsilon\}=x^*_\lambda\mathbb{N}=1$.

**Corollary 8.** *The inclusion $\mathcal{I}(ac_0)\oplus\{\lambda\mathbf{e}:\lambda\in\mathbb{R}\}\subseteq\mathcal{D}(ac_0)$ is proper.*

**Proof.** This result was obtained in [1]. Below we suggest the proof which, on the one hand, is more simple and, on the other hand, distinguishes a wide class of elements belonging to $\mathcal{D}(ac_0)$ while not representing in the form of $y+\lambda\mathbf{e}$ with $y\in\mathcal{I}(ac_0)$.

Let $\{d'_k\}$ and $\{d''_k\}$ be two sequences in $\mathbb{N}$ such that $d'_1=1$, $d'_k\le d''_k<d'_{k+1}$ for all $k$, and $\lim_{k\to\infty}(d''_k-d'_k)=\infty$. Then, in view of Corollary 7(a), the set $D=\bigcup_{k=1}^{\infty}[d'_{2k},d''_{2k}]\in\mathcal{D}(ac_0)$. On the other hand, using the relation (1), we have $\tau(\chi_D)=\tau(\chi_{\mathbb{N}\setminus D})=1$, whence there exist functionals $x^*_1,x^*_2\in$ BM satisfying $x^*_1D=1$ and $x^*_2D=0$. Now if $\chi_D=y+\lambda\mathbf{e}$ with $y\in\mathcal{I}(ac_0)$ and $\lambda\in\mathbb{R}$ then $x^*_iD=\lambda$ for $i=1,2$, a contradiction (the arguments above are also valid for the set $B=\bigcup_{k=1}^{\infty}[d'_{2k-1},d''_{2k-1}]$). $\square$

Part (g) of Theorem 3′ suggests the definition of the following sequence space

$$\mathcal{D}_0=\{z\in\ell_\infty:z-Tz\in c_0\}=\{z\in\ell_\infty:\lim_{n\to\infty}(z_{n+1}-z_n)=0\}$$

and a possible connection of it with Banach–Mazur limits. As is easy to see, $\mathcal{D}_0$ is a closed subalgebra of $\ell_\infty$ while is not an ideal in $\ell_\infty$. Evidently, $\mathcal{D}_0\subseteq\mathcal{D}(ac_0)$ and (see, e.g., [16, p. 139]) $\mathcal{D}_0\cap ac_0=c_0$. The space $\mathcal{D}'_0=\{z\in s:z-Tz\in c_0\}$ also can be considered, but $\mathcal{D}'_0\not\subseteq\ell_\infty$. We can go some more further and, for an arbitrary ideal $J$ in $\ell_\infty$, consider the space $\mathcal{D}_J=\{z:z-Tz\in J\}$. As far as the author knows, spaces $\mathcal{D}_J$ and, in particular, $\mathcal{D}_0$ have not explored in detail.

The next result tells us when a positive functional $z^* \in \mathcal{D}(ac_0)^*$ extends to a functional on $\ell_\infty$ which belongs to BM. By $\mathcal{E}(z^*)$, we shall denote the set

$$\mathcal{E}(z^*) = \{x^* \in \ell_\infty^* : x^* \geq 0 \text{ and } x^*z = z^*z \text{ for all } z \in \mathcal{D}(ac_0)\}.$$

**Theorem 9.** *For a functional $z^* \in S^+_{\mathcal{D}(ac_0)^*}$ the following statements are equivalent:*

(a) $\mathrm{BM} \cap \mathcal{E}(z^*) \neq \emptyset$;
(b) *The equality $z^*z = z^*(Tz)$ holds for all $z \in \mathcal{D}(ac_0)$;*
(c) $z^*(\mathcal{I}(ac_0)) = \{0\}$.

*If, in addition, $z^*$ is multiplicative on $\mathcal{D}(ac_0)$ then the statements* (a)–(c) *are equivalent to the next:*

(d) $(\mathrm{ext}\,\mathrm{BM}) \cap \mathcal{E}(z^*) \neq \emptyset$.

**Proof.** The implications (d) $\Longrightarrow$ (a) $\Longrightarrow$ (c) are obvious.

(b) $\Longrightarrow$ (a) By Kantorovič theorem [3, p. 26], $\mathcal{E}(z^*) \neq \emptyset$. Moreover, as is easy to see, the set $\mathcal{E}(z^*)$ is convex, $\sigma(\ell_\infty^*, \ell_\infty)$-compact, and $T^*$-invariant. Now the desired assertion follows immediately from Schauder–Tychonoff Fixed Point Theorem [2, p. 583] (the given argument is valid for every $T$-invariant subspace $\mathcal{D}$ of $\ell_\infty$ which contains $\mathbf{e}$).

(c) $\Longrightarrow$ (b) In view of Theorem 3′(g), for an arbitrary element $z \in \mathcal{D}(ac_0)$ the inclusion $z - Tz \in \mathcal{I}(ac_0)$ holds. Now the statement (b) is clear.

The proof of the implication (c) $\Longrightarrow$ (d) will be given in the next section. □

## 3. Banach–Mazur limits as measures on $\beta\mathbb{N}$

We recall first that a *compactification* [2, p. 56] of a (Hausdorff) topological space $X$ is a compact space $Y$ where $X$ is homeomorphic to a dense subset of $Y$, so we may treat $X$ as an actual dense subset of $Y$. A space $X$ has a compactification iff $X$ is completely regular. Moreover, in this case, $X$ has a compactification $\beta X$ with the following property: every continuous bounded real function $f$ on $X$ has a (unique) continuous extension $f^\beta$ from $\beta X$ to $\mathbb{R}$. Furthermore, $\beta X$ is unique, in the following sense: if a compactification $Y$ of $X$ satisfies this property then there exists a homeomorphism of $\beta X$ onto $Y$ that leaves $X$ pointwise fixed. This compactification $\beta X$ is called the *Stone–Čech compactification* of $X$ (see [7, Chapter 6] and [2, Sections 2.17, 2.18] for details). Obviously, the mapping $f \to f^\beta$ defines an isometric isomorphism from the space $C_b(X)$ of all continuous bounded functions on $X$ onto the space $C(\beta X)$ which preserves algebraic operations and lattice operations. We also mention the next properties of $\beta X$ which will be used below. A completely regular space $X$ is extremally disconnected, i.e., every pair of disjoint open subsets of $X$ have disjoint open closures, iff $\beta X$ is extremally disconnected [7, p. 96, Exercise 6M.1]. Next, for every infinite discrete space $X$ the identity [7, p. 130]

$$\mathrm{card}\,\beta X = 2^{2^{\mathrm{card}\,X}} \tag{12}$$

holds. Moreover, if $X$ is a Lindelöf space, i.e., every open cover has a countable subcover, and is also locally compact then [7, pp. 115, 133] the cardinality of every infinite closed subset of $\beta X \setminus X$ is at least $2^{\mathfrak{c}}$.

The set $\mathbb{N}$ with the discrete topology is a completely regular space and, hence, it has the Stone–Čech compactification $\beta\mathbb{N}$. In view of the remarks above, $\beta\mathbb{N}$ is extremally disconnected. Moreover, [7, p. 99, Exercise 6S.3] every open-and-closed subset of $\beta\mathbb{N}$ is of the form $\overline{A}^{\beta\mathbb{N}}$ for some $A \subseteq \mathbb{N}$ and the sets of this form constitute a base for the topology on $\beta\mathbb{N}$.

Every sequence $x \in \ell_\infty$ can be considered as a continuous bounded function on the set $\mathbb{N}$ with the discrete topology. Therefore, this function extends uniquely to the continuous function $\widehat{x}$ from $\beta\mathbb{N}$ to $\mathbb{R}$. The mapping $x \to \widehat{x}$ is a lattice isometry from $\ell_\infty$ onto $C(\beta\mathbb{N})$. Consequently, every functional on $C(\beta\mathbb{N})$ defines a functional on $\ell_\infty$. In particular, for every point $t \in \beta\mathbb{N}$ there exists a functional $x_t^* \in \ell_\infty^*$ such that $x_t^* x = \delta_t \widehat{x} = \widehat{x}(t)$ for all $x \in \ell_\infty$. A functional $x^* \in S^+_{\ell_\infty^*}$ can be represented in the form $x^* = x_t^*$ for some $t \in \beta\mathbb{N}$ iff $x^*$ is a lattice (algebraic) homomorphism on $\ell_\infty$. Moreover, for a point $t \in \beta\mathbb{N}$, we have the inclusion $t \in \widehat{\mathbb{N}} = \beta\mathbb{N} \setminus \mathbb{N}$ iff $x_t^*(c_0) = \{0\}$. Next, by Riesz Representation Theorem [2, p. 497], for every functional $x^* \in \ell_\infty^*$ there exists a unique regular signed Borel (countable additive) measure $\mu_{x^*}$ (of bounded variation) on $\beta\mathbb{N}$ satisfying

$$x^* x = \int_{\beta\mathbb{N}} \widehat{x}\, d\mu_{x^*} \tag{13}$$

for all $x \in \ell_\infty$. As usual, the *support* of a *measure* $\mu_{x^*}$ (see, e.g., [2, Section 12.3]) will be denoted by $\operatorname{supp} \mu_{x^*}$.

The operator $T$ on $\ell_\infty$ is lattice homomorphism. Therefore, for an arbitrary point $t \in \widehat{\mathbb{N}}$ the functional $T^* x_t^*$ is also a lattice homomorphism, whence $T^* x_t^* = x^*_{\varphi(t)}$ for some point $\varphi(t) \in \widehat{\mathbb{N}}$. Analogously, $U^* x_t^* = x^*_{\psi(t)}$ for some point $\psi(t) \in \widehat{\mathbb{N}}$. As is easy to see, the mappings $\varphi, \psi : \widehat{\mathbb{N}} \to \widehat{\mathbb{N}}$ which have been constructed satisfy $\varphi(\psi(t)) = \psi(\varphi(t)) = t$ for all $t \in \widehat{\mathbb{N}}$ and so $\varphi = \psi^{-1}$. Next, since a net $\{t_\alpha\}$ in $\beta\mathbb{N}$ converges to a point $t$ iff $x^*_{t_\alpha} \xrightarrow{\sigma(\ell_\infty^*, \ell_\infty)} x_t^*$, the mappings $\varphi$ and $\psi$ are continuous. Finally, $\varphi$ and $\psi$ are homeomorphisms.

It is well known (see, e.g., [8]) that a functional $x^* \in \ell_\infty^*$ is a Banach–Mazur limit iff $\mu_{x^*}$ is a probability measure, the inclusion $\operatorname{supp} \mu_{x^*} \subseteq \widehat{\mathbb{N}}$ holds, and $\mu_{x^*}(B) = \mu_{x^*}(\varphi(B))$ ($\mu_{x^*}(B) = \mu_{x^*}(\psi(B))$) for every Borel subset $B$ of $\widehat{\mathbb{N}}$. As is easy to see, the mapping $\psi$ extends to a continuous mapping from $\beta\mathbb{N}$ into $\beta\mathbb{N}$ via the formula $\psi(n) = n + 1$ for all $n \in \mathbb{N}$. In this case, for $x^* \in \ell_\infty^*$, we have $x^* \in \mathrm{BM}$ iff $\mu_{x^*}$ is a probability measure being $\psi$-invariant, i.e., $\mu_{x^*}(B) = \mu_{x^*}(\psi^{-1}(B))$ for every Borel subset $B$ of $\beta\mathbb{N}$.

**Theorem 10.** *Let* $x^* \in \mathrm{BM}$. *Then* $\operatorname{supp} \mu_{x^*} = \varphi(\operatorname{supp} \mu_{x^*}) = \psi(\operatorname{supp} \mu_{x^*})$.

**Proof.** We will verify the *$\varphi$-invariance* of the set $\operatorname{supp} \mu_{x^*}$, i.e., the validity of the inclusion $\varphi(\operatorname{supp} \mu_{x^*}) \subseteq \operatorname{supp} \mu_{x^*}$. Proceeding by contradiction, we find a point

$$t \in \operatorname{supp} \mu_{x^*} \tag{14}$$

such that $\varphi(t) \notin \operatorname{supp} \mu_{x^*}$. There exists a neighborhood $\mathcal{U}_{\varphi(t)}$ of the point $\varphi(t)$ satisfying $\mathcal{U}_{\varphi(t)} \cap \operatorname{supp} \mu_{x^*} = \emptyset$. By Urysohn's lemma [7, p. 44], for some element $x \in \ell_\infty^+$, we have $\widehat{x}(\varphi(t)) > 0$ and $\operatorname{supp} \widehat{x} \subseteq \mathcal{U}_{\varphi(t)}$. Then, on the one hand, $0 = x^* x = x^*(Tx)$, whence

$$\operatorname{supp} \widehat{Tx} \cap \operatorname{supp} \mu_{x^*} = \emptyset, \tag{15}$$

on the other hand, $0 < \widehat{x}(\varphi(t)) = \delta_{\varphi(t)} \widehat{x} = \delta_t(\widehat{Tx})$, whence $t \in \operatorname{supp} \widehat{Tx}$. The latter contradicts (14) and (15). The inclusion $\psi(\operatorname{supp} \mu_{x^*}) \subseteq \operatorname{supp} \mu_{x^*}$ can be checked by a similar manner. Now the required assertion is obvious. $\square$

For an arbitrary subset $D$ of $\beta\mathbb{N}$, we define the subspace $L_D$ of $\ell_\infty^*$ as the span of the set $\{x_t^* : t \in D\}$ (if $D = \emptyset$, we put $L_D = \{0\}$). Since for every $t \in \beta\mathbb{N}$ the functional $x_t^*$ is an atom in the Banach lattice $\ell_\infty^*$, the subspace $L_D$ is an ideal in $\ell_\infty^*$.

**Lemma 11.** *For a subset $D$ of $\widehat{\mathbb{N}}$ the following statements hold:*

(a) *If $D$ is $\varphi$ (or $\psi$)-invariant then the bipolar $L_D^{\circ\circ}$ and the set $L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$ is $T^*$ (or $U^*$)-invariant;*
(b) *If $\varphi(D) = D$ then*

$$T^* L_D^{\circ\circ} = U^* L_D^{\circ\circ} = L_D^{\circ\circ} \quad \text{and} \quad T^*(L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+) = U^*(L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+) = L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+.$$

**Proof.** (a) Let $D$ be $\varphi$-invariant (the case of $\psi$ is analogous). The bipolar $L_D^{\circ\circ}$ was taken, of course, with respect to the dual system $\langle \ell_\infty, \ell_\infty^* \rangle$. By the Bipolar Theorem [3, p. 140], $L_D^{\circ\circ}$ is the $\sigma(\ell_\infty^*, \ell_\infty)$-closure of $L_D$. On the other hand, since the set $D$ is $\varphi$-invariant, the subspace $L_D$ is $T^*$-invariant and, hence, $L_D^{\circ\circ}$ is also $T^*$-invariant. Using the inclusion $D \subseteq \widehat{\mathbb{N}}$, we get the $T^*$-invariance of $L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$.

(b) For an arbitrary functional $x^* \in L_D^{\circ\circ}$ there exists a net $\{x_\alpha^*\}$ in the space $L_D$ such that $x_\alpha^* \xrightarrow{\sigma(\ell_\infty^*, \ell_\infty)} x^*$. Using our condition, we find a net $\{y_\alpha^*\}$ in $L_D$ satisfying $T^* y_\alpha^* = x_\alpha^*$. Consequently, $y_\alpha^* = U^* T^* y_\alpha^* \to U^* x^*$ and so $U^* x^* \in L_D^{\circ\circ}$, whence $x^* = T^* U^* x^* \in T^*(L_D^{\circ\circ})$ and so $L_D^{\circ\circ} \subseteq T^*(L_D^{\circ\circ})$. According to part (a), we have $L_D^{\circ\circ} = T^* L_D^{\circ\circ}$. The equality $U^* L_D^{\circ\circ} = L_D^{\circ\circ}$ follows from the identity $U^* T^* = I$ on the band $\ell_1^{\mathrm{d}}$ in the Banach lattice $\ell_\infty^*$.

Now let $x^* \in L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$. As showed above, there exists a functional $y^* \in L_D^{\circ\circ}$ such that $T^* y^* = x^*$. Since $T$ is an interval preserving operator, the adjoint operator $T^*$ is [3, p. 92] a lattice homomorphism, whence $T^* |y^*| = x^* = T^* y^*$. Therefore, $T^*(y^*)^- = 0$ and so $(y^*)^- = 0$. Thus, $y^* \geq 0$. Then $1 = \|x^*\|_{\ell_\infty^*} = y^*(T\mathbf{e}) = y^*\mathbf{e} = \|y^*\|_{\ell_\infty^*}$. Finally, $T^*(L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+) = L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$. The case of $U^*$ is analogous. □

**Theorem 12.** *For every nonempty $\varphi$ (or $\psi$)-invariant subset $D$ of $\widehat{\mathbb{N}}$, we have the relation* $(\operatorname{ext} \mathrm{BM}) \cap L_D^{\circ\circ} \neq \emptyset$.

**Proof.** Let $D$ be $\varphi$-invariant. In view of part (a) of the preceding lemma, the convex $\sigma(\ell_\infty^*, \ell_\infty)$-compact set $L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$ is $T^*$-invariant. Consequently, by Schauder–Tychonoff Fixed Point Theorem [2, p. 583], the set $N(I - T^*) \cap L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+$ is nonempty. Moreover, this set is also convex $\sigma(\ell_\infty^*, \ell_\infty)$-compact. By Krein–Milman theorem [3, p. 137], it has an extreme point

$$z^* \in \operatorname{ext}\,(N(I - T^*) \cap L_D^{\circ\circ} \cap S_{\ell_\infty^*}^+). \tag{16}$$

Clearly, $z^* \in \mathrm{BM}$. We claim that $z^* \in \operatorname{ext} \mathrm{BM}$. To this end, let $z^* = \frac{z_1^* + z_2^*}{2}$ with $z_1^*, z_2^* \in \mathrm{BM}$. Since $L_D$ is an ideal in the Banach lattice $\ell_\infty^*$, the bipolar $L_D^{\circ\circ}$ is a band and, in particular, is an ideal in $\ell_\infty^*$. Consequently, the last equality implies the inclusion $z_1^*, z_2^* \in L_D^{\circ\circ}$. Thus, in view of (16), we have $z_1^* = z_2^*$ and, hence, $z^* \in \operatorname{ext} \mathrm{BM}$. □

**Lemma 13.** *For $x^* \in \ell_\infty^*$ and a subset $D$ of $\beta\mathbb{N}$, we have the next statements:*

(a) *The relation $x^* \in L_D^{\circ\circ}$ holds iff $\mu_{|x^*|}(\overline{D}) = \mu_{|x^*|}(\beta\mathbb{N})$, i.e.,* $\operatorname{supp} \mu_{x^*} \subseteq \overline{D}$;
(b) *The relation $x^* \perp L_D^{\circ\circ}$ holds iff $\mu_{|x^*|}(\overline{D}) = 0$.*

**Proof.** (a) For the necessity, consider a point $t \in \beta\mathbb{N}$ which does not belong to the closure $\overline{D}$ of $D$. Pick a neighborhood $\mathcal{U}_t$ of $t$ such that $\mathcal{U}_t \cap \overline{D} = \emptyset$. By Urysohn's lemma [7, p. 44], there exists an element $x \in \ell_\infty^+$ satisfying $\widehat{x}(t) > 0$ and $\operatorname{supp} \widehat{x} \subseteq \mathcal{U}_t$. Using our condition, we find a net $\{x_\alpha^*\}$ in $L_D$ with the property $x_\alpha^* \xrightarrow{\sigma(\ell_\infty^*, \ell_\infty)} x^*$. Then $0 = x_\alpha^* x \to x^* x$ and so $x^* x = 0$, whence $t \notin \operatorname{supp} \mu_{x^*}$.

For the converse, proceeding by contradiction and using the classical Separation Theorem [3, p. 136], we find an element $y \in \ell_\infty$ satisfying $x^*y \neq 0$ and $y^*y = 0$ for all $y^* \in L_D^{\circ\circ}$. Thus, $\widehat{y}(s) = 0$ for all $s \in \overline{D}$. Consequently, we obtain $x^*y = \int_{\overline{D}} \widehat{y}\, d\mu_{x^*} = 0$, a contradiction.

(b) Let $x^* \perp L_D^{\circ\circ}$. Define the measure $\mu_{x^*}^D$ on the $\sigma$-algebra $\mathcal{B}(\beta\mathbb{N})$ of all Borel sets of $\beta\mathbb{N}$ via the formula $\mu_{x^*}^D(B) = \mu_{|x^*|}(\overline{D} \cap B)$ for all $B \in \mathcal{B}(\beta\mathbb{N})$. As is easy to see, $\mu_{x^*}^D$ is regular. Thus, for some $y^* \in \ell_\infty^*$, we have $\mu_{x^*}^D = \mu_{y^*}$. Moreover, the inclusion supp $\mu_{y^*} \subseteq \overline{D}$ holds. Whence, in view of part (a), we get $y^* \in L_D^{\circ\circ}$. On the other hand, using the inequalities $\mu_{|x^*|} \geq \mu_{y^*} \geq 0$, we have $|x^*| \geq y^* \geq 0$. Therefore, $y^* \perp L_D^{\circ\circ}$. Finally, $y^* = 0$ and so $\mu_{|x^*|}(\overline{D}) = 0$.

For the converse, consider an arbitrary functional $z^* \in [0, |x^*|] \cap L_D^{\circ\circ}$. Then, on the one hand, the inequality $\mu_{z^*} \leq \mu_{|x^*|}$ implies $\mu_{z^*}(\overline{D}) = 0$, on the other hand, supp $\mu_{z^*} \subseteq \overline{D}$, whence $\mu_{z^*} = 0$ and so $z^* = 0$. Finally, $x^* \perp L_D^{\circ\circ}$. □

The preceding lemma is valid for an arbitrary subset $D$ of some compact space $K$, a functional $x^* \in C(K)^*$, and the span $L_D$ of $\{\delta_d : d \in D\}$.

From Theorem 12 and Lemma 13(a), the next result [4] follows.

**Corollary 14.** *For every nonempty $\varphi$ (or $\psi$)-invariant subset $D$ of $\widehat{\mathbb{N}}$ there exists a functional $z^* \in$ ext BM satisfying* supp $\mu_{z^*} \subseteq \overline{D}$.

As is well known, every closed (algebraic or order) ideal $J$ in the space $C(K)$, where $K$ is compact, can be represented in the form $J = \{x \in C(K) : x(A) = \{0\}\}$ for some closed subset $A$ of $K$; the converse is obvious. Consequently, the ideal $\mathcal{I}(ac_0)$ in the space $\ell_\infty$ can be represented in the form $\mathcal{I}(ac_0) = \{x \in \ell_\infty : \widehat{x}(\mathbb{A}) = \{0\}\}$ for some closed subset $\mathbb{A}$ of $\beta\mathbb{N}$. As is easy to see, $\mathbb{A} \subsetneq \widehat{\mathbb{N}}$.

Next, the *orbit* (with respect to the mapping $\varphi$ or $\psi$) of a point $t \in \widehat{\mathbb{N}}$ is the set

$$\mathcal{O}_t = \{\varphi^n(t) : n \in \mathbb{Z}\} = \{\psi^n(t) : n \in \mathbb{Z}\}.$$

It is not difficult to show that $\varphi(\overline{\mathcal{O}_t}) = \psi(\overline{\mathcal{O}_t}) = \overline{\mathcal{O}_t}$.

**Theorem 15.** *The following statements hold:*

(a) *The equalities $\varphi(\mathbb{A}) = \psi(\mathbb{A}) = \mathbb{A}$ are valid and, in particular, for every point $t \in \mathbb{A}$, we have $\overline{\mathcal{O}_t} \subseteq \mathbb{A}$;*
(b) $\mathbb{A} = \overline{\bigcup_{x^* \in \mathrm{BM}} \mathrm{supp}\mu_{x^*}} = \overline{\bigcup_{x^* \in \mathrm{extBM}} \mathrm{supp}\mu_{x^*}}$*;*
(c) *For every nonempty $\varphi$ (or $\psi$)-invariant subset $D$ of $\widehat{\mathbb{N}}$ the relation $\overline{D} \cap \mathbb{A} \neq \emptyset$ is valid and, in particular, for every point $t \in \widehat{\mathbb{N}}$, we have $\overline{\mathcal{O}_t} \cap \mathbb{A} \neq \emptyset$;*
(d) *For an arbitrary element $z \in \mathcal{D}(ac_0)$ and a point $t \in \mathbb{A}$ the function $\widehat{z}$ is constant on the set $\overline{\mathcal{O}_t}$.*

**Proof.** (a) Consider an arbitrary point $s \in \mathbb{A}$ and an element $x \in \mathcal{I}(ac_0)$. We have the equalities $x^*_{\varphi(s)}x = (T^*x^*_s)x = x^*_s(Tx) = 0$ and, hence, $\varphi(s) \in \mathbb{A}$. Therefore, $\varphi(\mathbb{A}) \subseteq \mathbb{A}$. Analogously, $\psi(\mathbb{A}) \subseteq \mathbb{A}$. Finally, $\varphi(\mathbb{A}) = \psi(\mathbb{A}) = \mathbb{A}$.

(b) The inclusions $\overline{\bigcup_{x^* \in \mathrm{ext\ BM}} \mathrm{supp}\mu_{x^*}} \subseteq \overline{\bigcup_{x^* \in \mathrm{BM}} \mathrm{supp}\mu_{x^*}} \subseteq \mathbb{A}$ are obvious. We verify the inclusion $\mathbb{A} \subseteq \mathbb{A}' = \overline{\bigcup_{x^* \in \mathrm{ext\ BM}} \mathrm{supp}\mu_{x^*}}$. To this end, we consider an arbitrary point $s \notin \mathbb{A}'$ and find a neighborhood $\mathcal{U}_s$ of $s$ such that $\mathcal{U}_s \cap \mathbb{A}' = \emptyset$. By Urysohn's lemma [7, p. 44], there exists an element $y \in \ell_\infty^+$ satisfying $\widehat{y}(s) > 0$ and supp $\widehat{y} \subseteq \mathcal{U}_s$. Therefore, $x^*y = 0$ for all $x^* \in$ ext BM and so $y \in ac_0$. Thus, we obtain $y \in \mathcal{I}(ac_0)$ as $y \geq 0$. Finally, $s \notin \mathbb{A}$.

(c) This statement follows at once from Corollary 14 and part (b) above.

(d) In view of Theorem 3′(g), we have $0 = x_t^*(z - Tz) = \widehat{z}(t) - \widehat{z}(\varphi(t))$. Whence, using part (a) above, we obtain easily the identity $\widehat{z}(\varphi^n(t)) = \widehat{z}(t)$ for all $n \in \mathbb{Z}$. Thus, $\widehat{z}(w) = \widehat{z}(t)$ for all $w \in \overline{\mathcal{O}_t}$. □

The set $\overline{\bigcup_{x^* \in \mathrm{BM}} \operatorname{supp} \mu_{x^*}}$ was earlier considered, e.g., in [4] (where it was denoted by $K^\tau$), but from another viewpoint.

Now we are ready to prove the implication (c) $\Longrightarrow$ (d) of Theorem 9. To this end, let $z^*$ be a positive multiplicative functional on $\mathcal{D}(ac_0)$ such that $z^*\mathbf{e} = 1$ and $z^*(\mathcal{I}(ac_0)) = \{0\}$. Since $\mathcal{D}(ac_0)$ is a closed subalgebra of $\ell_\infty$, $z^*$ is a lattice homomorphism. Therefore, by Lipecki–Luxemburg–Schep theorem [3, p. 99], $z^*$ extends to all of $\ell_\infty$ as a lattice homomorphism. Thus, there exists a point $t \in \beta\mathbb{N}$ satisfying the relation

$$x_t^* z = z^* z \tag{17}$$

for all $z \in \mathcal{D}(ac_0)$. The inclusion $t \in \mathbb{A}$ holds as $z^*(\mathcal{I}(ac_0)) = \{0\}$. In view of part (d) of the preceding theorem, $z^*z = \widehat{z}(s)$ for all $s \in \overline{\mathcal{O}_t}$. On the other hand, Theorem 12 guarantees the existence of a functional $x^* \in (\operatorname{ext} \mathrm{BM}) \cap L^{\infty}_{\overline{\mathcal{O}_t}}$. A glance at Lemma 13(a) yields the relation $\operatorname{supp} \mu_{x^*} \subseteq \overline{\mathcal{O}_t}$ and, hence, $x^*z = \int_{\overline{\mathcal{O}_t}} \widehat{z}\, d\mu_{x^*} = \widehat{z}(t) = z^*z$ for all $z \in \mathcal{D}(ac_0)$. Thus, $x^* = z^*$ on $\mathcal{D}(ac_0)$, i.e., $x^* \in \mathcal{E}(z^*)$. □

**Lemma 16.** *If the identity $T^*x^* = x^*$ holds with $x^* \in \ell_\infty^*$ then $T^*B_{x^*} = B_{x^*}$.*

**Proof.** First of all, we recall that $B_{x^*}$ is the band generated by $x^*$ in the Banach lattice $\ell_\infty^*$. Next, the operator $T$ is one-to-one with closed range $R(T)$, whence $R(T^*) = \ell_\infty^*$.

Using the identities $T^*|x^*| = |x^*|$ and $B_{|x^*|} = B_{x^*}$, we can assume $x^* \geq 0$. Evidently, $T^*B_{x^*} \subseteq B_{x^*}$. Let us verify the converse inclusion. To this end, let $y^* \in B_{x^*}$. In view of the remarks above, there exists a functional $z^* \in \ell_\infty^*$ satisfying $T^*z^* = y^*$. Clearly, $z^*$ can be represented in the form $z^* = z_1^* + z_2^*$, where $z_1^* \in B_{x^*}$ and $z_2^* \perp B_{x^*}$. Since $z_2^* \perp x^*$ and $T^*$ is a lattice homomorphism, we have $T^*z_2^* \perp x^*$. Whence, using the equality $y^* = T^*z_1^* + T^*z_2^*$, we obtain $y^* = T^*z_1^* \in T^*B_{x^*}$. □

Now we are ready to derive the next characterization of extreme points of BM.

**Theorem 17.** *For a functional $z^* \in \mathrm{BM}$ the following statements are equivalent:*

(a) $z^* \in \operatorname{ext} \mathrm{BM}$;
(b) *For every band $B$ in $\ell_\infty^*$ such that $T^*B = B$, we have either $z^* \in B$ or $z^* \perp B$;*
(c) *For every principal band $B_{y^*}$ in $\ell_\infty^*$, where $y^* \in \ell_\infty^*$, such that $T^*B_{y^*} = B_{y^*}$, we have either $z^* \in B_{y^*}$ or $z^* \perp B_{y^*}$.*

**Proof.** (a) $\Longrightarrow$ (b) The element $z^*$ can be represented in the form

$$z^* = z_1^* + z_2^*, \tag{18}$$

where $z_1^* \in B$ and $z_2^* \perp B$. Evidently, $z_i^* \geq 0$ and $T^*z_1^* \in B$. Moreover,

$$T^*z_2^* \perp B. \tag{19}$$

Indeed, if $x^* \in B$ then $T^*x_0^* = x^*$ for some $x_0^* \in B$, whence $(T^*z_2^*) \wedge |x^*| = T^*(z_2^* \wedge |x_0^*|) = 0$ and (19) has been checked. Using (18), we have $z_1^* + z_2^* = z^* = T^*z^* = T^*z_1^* + T^*z_2^*$. Thus, in view of (19), $z_i^* = T^*z_i^*$. If $z_i^* > 0$ for $i = 1, 2$ then, on the one hand, the last equality implies $\frac{z_i^*}{\|z_i^*\|_{\ell_\infty^*}} \in \mathrm{BM}$, on the other hand,

$$z^* = \|z_1^*\|_{\ell_\infty^*} \frac{z_1^*}{\|z_1^*\|_{\ell_\infty^*}} + \|z_2^*\|_{\ell_\infty^*} \frac{z_2^*}{\|z_2^*\|_{\ell_\infty^*}} \in \operatorname{ext} \mathrm{BM}.$$

Consequently, $\frac{z_1^*}{\|z_1^*\|_{\ell_\infty^*}} = \frac{z_2^*}{\|z_2^*\|_{\ell_\infty^*}}$, which is impossible. Therefore, either $z_1^* = 0$ or $z_2^* = 0$.

The implication (b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a) As was mentioned in Section 1, the inclusion $z^* \in \operatorname{ext} \mathrm{BM}$ holds iff $z^*$ is an atom in the $AL$-space $N(I - T^*)$. Consequently, assuming $z^* \notin \operatorname{ext} \mathrm{BM}$, we find non-zero functionals $y_1^*$ and $y_2^*$ satisfying $y_1^* \perp y_2^*$, $T^*y_i^* = y_i^*$, and $y_1^* + y_2^* = z^*$, in particular, $z^* \notin B_{y_i^*}$ for $i = 1, 2$. In view of the preceding lemma, $T^*B_{y_i^*} = B_{y_i^*}$. Whence, using our condition, we infer $z^* \in B_{y_1^*}^{\mathrm{d}} \cap B_{y_2^*}^{\mathrm{d}}$ and, hence, $z^* = 0$, a contradiction. □

The assertions which are analogous to Lemma 16 and Theorem 17 also hold for the case of the operator $U$.

**Corollary 18.** *Let $z^* \in \operatorname{ext} \mathrm{BM}$. Then for every subset $D$ of $\widehat{\mathbb{N}}$ such that $\varphi(D) = D$, we have either $z^* \in L_D^{\circ\circ}$, i.e., $\mu_{z^*}(\overline{D}) = 1$, or $z^* \perp L_D^{\circ\circ}$, i.e., $\mu_{z^*}(\overline{D}) = 0$.*

The preceding corollary follows at once from Theorem 17(b), Lemma 11(b), and Lemma 13. As a matter of fact, in view of Lemma 13, it is also a simple consequence of well-known results about ergodic measures (see, e.g., [2, Section 19.5]). Nevertheless, Theorem 17 allows us to look at these results from the viewpoint of the theory of ordered linear spaces. Unfortunately, the converse to Corollary 18 is false, i.e., there exists a functional $x^* \in \mathrm{BM}$ such that the relation $\mu_{x^*}(D) \in \{0, 1\}$ holds for every closed subset $D$ of $\beta\mathbb{N}$ satisfying $\varphi(D) = D$ while $x^* \notin \operatorname{ext} \mathrm{BM}$. In fact, there exists ([12]; see also remarks in the next section) a functional $x^* \in \mathrm{BM} \setminus \operatorname{ext} \mathrm{BM}$ such that supp $\mu_{x^*}$ is a minimal (by inclusion) closed $\varphi$-invariant subset of $\widehat{\mathbb{N}}$. Consequently, if $\varphi(D) = D$ for a closed subset $D$ of $\widehat{\mathbb{N}}$ then either supp $\mu_{x^*} \subseteq D$, i.e., $\mu_{x^*}(D) = 1$, or supp $\mu_{x^*} \cap D = \emptyset$, i.e., $\mu_{x^*}(D) = 0$.

## 4. Some cardinalities

This section is devoted to the discussion of the question about the cardinality of some subsets of BM and of some notions which are closed to it.

Recall that $\ell_\infty^*$ is an $AL$-space, i.e., [3, p. 187] $\|x^* + y^*\|_{\ell_\infty^*} = \|x^*\|_{\ell_\infty^*} + \|y^*\|_{\ell_\infty^*}$ for all positive functionals $x^*, y^* \in \ell_\infty^*$. By Kakutani–Bohnenblust–Nakano theorem [3, p. 192], $\ell_\infty^*$ is lattice isometric onto a space $L_1(\Omega)$ of all integrable functions on some set $\Omega$ with the measure $\mu$. In it turn, the set $\Omega$ can be represented in the form of a disjoint union of measurable subsets $\Omega_d$ and $\Omega_c$ such that the measure $\mu$ on the first one is purely atomic and on the second one is nonatomic. Therefore, the representation

$$\ell_\infty^* = L_1(\Omega) = L_1(\Omega_d) \oplus L_1(\Omega_c) \tag{20}$$

holds, where the bands $L_1(\Omega_d)$ and $L_1(\Omega_c)$ are disjoint. A function $x \in L_1(\Omega)$ is an extreme point of the positive part $S_{L_1(\Omega)}^+$ of a unit sphere iff it can be represented in the form $x = \frac{\chi_A}{\mu(A)}$, where $A$ is an atom of the measure $\mu$. Thus, there is a one-to-one correspondence between the set ext $S_{L_1(\Omega)}^+$ and the set of atoms of $\mu$ on $\Omega$. In it turn, there is a one-to-one correspondence between the set ext $S^+_{L_1(\Omega)}$ and the set ext $S^+_{C(\beta\mathbb{N})^*}$ which is the collection of algebraic homomorphisms on $C(\beta\mathbb{N})$ and, hence, can be identified with $\beta\mathbb{N}$. Using the identity card $(\beta\mathbb{N}) = 2^{\mathfrak{c}}$ (see (12)), we have

$$\text{card}\,(\text{ext}\,S^+_{L_1(\Omega)}) = \text{card}\,(\text{atoms of }\mu) = 2^{\mathfrak{c}}.$$

On the other hand, there is a one-to-one correspondence between the set of the atoms of $\mu$ on $\Omega$ and the set of atoms in the Riesz space $L_1(\Omega_d)$ which is a maximal (by inclusion) disjoint system (md-system) in $L_1(\Omega_d)$ (the existence of such system in an arbitrary Riesz space $E$ is a simple consequence of Zorn's lemma). Since in an arbitrary Banach lattice $F$ with order continuous norm and, in particular, in an $AL$-space, every two infinite md-system have the same cardinality, we obtain that the cardinality of an md-system in $L_1(\Omega_d)$ is equal to $2^{\mathfrak{c}}$. The cardinality of an md-system in $L_1(\Omega_c)$ is also equal to $2^{\mathfrak{c}}$ (see remarks below). Since ext $S^+_{L_1(\Omega_c)} = \emptyset$, the band $L_1(\Omega_c)$ is not $\sigma(\ell^*_\infty, \ell_\infty)$-closed.

Next, the space $N(I - T^*)$ is (see, e.g., [1]) a Riesz subspace of $\ell^*_\infty$ and, hence, is also an $AL$-space under the norm and the order induced by $\ell^*_\infty$. Therefore, extreme points of the set BM are also pairwise disjoint in $\ell^*_\infty$. Analogously, we have the representation

$$N(I - T^*) = L_1(\Omega^{\text{BM}}) = L_1(\Omega^{\text{BM}}_d) \oplus L_1(\Omega^{\text{BM}}_c), \tag{20$'$}$$

where the set $\Omega^{\text{BM}}$ with the measure $\mu^{\text{BM}}$ satisfies $\Omega^{\text{BM}} = \Omega^{\text{BM}}_d \cup \Omega^{\text{BM}}_c$ and $\mu^{\text{BM}}$ on $\Omega^{\text{BM}}_d$ is purely atomic and on $\Omega^{\text{BM}}_c$ is nonatomic. It should be noted that $L_1(\Omega^{\text{BM}}_d)$ and $L_1(\Omega^{\text{BM}}_c)$ are bands in $N(I - T^*)$ while are not bands in $\ell^*_\infty$.

As was shown in [5] (see also Theorem 21(a)),

$$\text{card}\,(\text{ext BM}) = \text{card}\,(\text{atoms of }\mu^{\text{BM}}) = 2^{\mathfrak{c}}.$$

Thus, in view of the relation $N(I-T^*) \perp L_1(\Omega_d)$ (see, e.g., [1]), the cardinality of an md-system in $L_1(\Omega_c)$ is equal to $2^{\mathfrak{c}}$. There exists ([4]; see also remarks after the proof of Theorem 21) a functional $x^* \in$ BM satisfying $x^* \perp$ ext BM and so $\mu^{\text{BM}}(\Omega^{\text{BM}}_c) > 0$, i.e., measure $\mu^{\text{BM}}$ on $\Omega^{\text{BM}}$ is not purely atomic. As a matter of fact, it will be shown below (see Theorem 21(b)) that the cardinality of an md-system in $L_1(\Omega^{\text{BM}}_c)$ is also equal $2^{\mathfrak{c}}$ and, in particular, the restriction $\mu^{\text{BM}}_c$ of $\mu^{\text{BM}}$ on $\Omega^{\text{BM}}_c$ is not a $\sigma$-finite measure.

Let us consider two operators $Q_1$ and $Q_2$ on $\ell_\infty$ defined by

$$Q_1 x = \Big(x_1, \frac{x_2 + x_3}{2}, \frac{x_3 + x_4 + x_5}{3}, \dots\Big) \quad \text{and} \quad Q_2 x = (x_1, x_2, x_2, x_3, x_3, x_3, \dots).$$

Obviously, $Q_1 Q_2 = I$ and $Q^*_2 Q^*_1 = I$. Since the range $R(Q_1) = \ell_\infty$, $Q^*_1$ is one-to-one. Using the inclusion $Q_1(bs) \subseteq c_0$, we have $Q^*_1(\ell^{\text{d}}_1) \subseteq N(I - T^*)$ and $Q^*_1(S^+_{\ell^{\text{d}}_1}) \subseteq$ BM. As is easy to see, the operator $Q_1$ is interval preserving and, hence, [3, p. 92] $Q^*_1$ is lattice homomorphism. Therefore, for any md-system $\{y^*_\alpha\}$ in the band $\ell^{\text{d}}_1$ of $\ell^*_\infty$, the collection $\{Q^*_1 y^*_\alpha\}$ is a disjont system in $L_1(\Omega^{\text{BM}})$. In particular, $\{Q^*_1 x^*_t : t \in \widehat{\mathbb{N}}\}$ is a disjoint system in BM.

Next, fix a natural number $k \geq 2$ and define the subsets $D^k_1, \dots, D^k_k$ of $\mathbb{N}$ as follows. For $i = 1, \dots, k$ and $j \in \mathbb{N}$, we put

$$D^k_{ij} = \Big[\frac{(j-1)j}{2}k + (i-1)j + 1, \frac{(j-1)j}{2}k + ij\Big].$$

Evidently, for every $i$ the subsets $D_{i1}^k, D_{i2}^k, \dots$ of $\mathbb{N}$ are pairwise disjoint. Now, let $D_i^k = \bigcup_{j=1}^{\infty} D_{ij}^k$. For instance, for $k = 2$, we have

$$\chi_{D_1^2} = (1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, \dots) \quad \text{and}$$
$$\chi_{D_2^2} = (0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, \dots).$$

In view of Corollary 7(a), $D_i^k \in \mathcal{D}(ac_0)$ for all $i = 1, \dots, k$.

Let us also define the subsets $A_n$ of $\mathbb{N}$ by $A_n = [\frac{(n-1)n}{2} + 1, \frac{n(n+1)}{2}]$, where $n \in \mathbb{N}$. The next results will be needed latter.

**Lemma 19.** *Let $k \in \mathbb{N}$ and let $i \in \{1, \dots, k\}$. If $k > 2$ then for every $n \in \mathbb{N}$ there exists at most one index $j \in \mathbb{N}$ such that $A_n \cap D_{ij}^k \neq \emptyset$.*

**Proof.** Proceeding by contradiction, we find an index $j_0$ satisfying the relations $A_n \cap D_{ij_0}^k \neq \emptyset$ and $A_n \cap D_{i,j_0+1}^k \neq \emptyset$. Therefore, we have

$$\begin{aligned} \frac{(n-1)n}{2} + 1 &\le \frac{(j_0-1)j_0}{2}k + ij_0 \\ &\le \frac{j_0(j_0+1)}{2}k + (i-1)(j_0+1) + 1 \le \frac{n(n+1)}{2}. \end{aligned} \tag{21}$$

Hence, using the identity $n = \frac{n(n+1)}{2} - \left(\frac{(n-1)n}{2} + 1\right) + 1$, we get $n \ge j_0(k-1) + i + 1$. Taking into account the first inequality in (21) once more, we get $\frac{(j_0-1)j_0}{2}k + ij_0 \ge \frac{(j_0(k-1)+i)(j_0(k-1)+i+1)}{2} + 1$ and so

$$(j_0-1)j_0k + 2ij_0 \ge (j_0(k-1)+i)^2 + j_0(k-1) + i + 2. \tag{22}$$

Since $k > 2$, the inequalities $(j_0-1)j_0k < j_0^2(k-1)^2$ and $2ij_0 < 2ij_0(k-1)$ hold, which contradicts (22). $\square$

The preceding lemma does not hold in the case of $k = 2$.

**Lemma 20.** *For every $k \in \mathbb{N}$, $k > 2$, and $i \in \{1, \dots, k\}$ the inequality $\limsup_{n\to\infty}(Q_1\chi_{D_i^k})_n \le \frac{1}{\sqrt{k}}$ holds.*

**Proof.** First of all, we mention the identity $(Q_1x)_n = \frac{1}{n}\sum_{j\in A_n} x_j$ which is valid for all $x \in \ell_\infty$ and $n \in \mathbb{N}$. Therefore, if $A_n \cap D_i^k = \emptyset$ then $(Q_1\chi_{D_i^k})_n = 0$. Now assume that the set $N = \{n \in \mathbb{N} : A_n \cap D_i^k \neq \emptyset\}$ is infinite. In view of Lemma 19, for every $n \in N$ there exists a unique index $j_n$ satisfying $A_n \cap D_{ij_n}^k \neq \emptyset$ and, in particular, $A_n \cap D_{i,j_n+1}^k = \emptyset$. Thus,

$$\frac{(j_n-1)j_n}{2}k + (i-1)j_n + 1 \le \frac{n(n+1)}{2} \le \frac{j_n(j_n+1)}{2}k + (i-1)(j_n+1) + 1. \tag{23}$$

The second inequality in (23) implies

$$\limsup_{\substack{n\to\infty \\ n\in N}} j_n = \infty. \tag{24}$$

On the other hand, from the first inequality in (23), we obtain the inequality $n^2 + n - m_n \ge 0$, where $m_n = (j_n-1)j_nk + 2(i-1)j_n + 2$, and, hence, $n \ge \frac{-1+\sqrt{1+4m_n}}{2}$. Consequently,

$(Q_1\chi_{D_i^k})_n \le \frac{j_n}{n} \le \frac{2j_n}{-1+\sqrt{1+4m_n}}$. Finally, using (24), we infer

$$\limsup_{\substack{n\to\infty\\ n\in N}}(Q_1\chi_{D_i^k})_n \le \lim_{\substack{n\to\infty\\ n\in N}} \frac{2j_n}{-1+\sqrt{1+4m_n}} = \frac{1}{\sqrt{k}},$$

and the proof is complete. $\square$

As can be shown, the preceding lemma also holds in the case of $k = 2$.

Now we are ready to prove the main result of this section.

**Theorem 21.** *The following statements hold:*

(a) *An md-system in $L_1(\Omega_d^{\mathrm{BM}})$ has the cardinality $2^{\mathfrak{c}}$ and, in particular,* card (ext BM) $= 2^{\mathfrak{c}}$*;*
(b) *An md-system in $L_1(\Omega_c^{\mathrm{BM}})$ has the cardinality $2^{\mathfrak{c}}$;*
(c) *An md-system in the band $(L_1(\Omega_d) \oplus L_1(\Omega^{\mathrm{BM}}))^{\mathrm{d}}$ has the cardinality $2^{\mathfrak{c}}$.*

**Proof.** (a) We mentioned earlier that this statement was established in [5][1] (see also the identity (26)). We will suggest another proof.

For every point $t \in \widehat{\mathbb{N}}$, we define the set $\mathrm{BM}_t = \{x^* \in \mathrm{BM} : Q_2^* x^* = x_t^*\}$. In view of the remarks above, the set $\mathrm{BM}_t$ is nonempty. Moreover, $\mathrm{BM}_t$ is convex and $\sigma(\ell_\infty^*, \ell_\infty)$-compact. By Krein–Milman theorem [3, p. 137], ext $\mathrm{BM}_t \neq \emptyset$. The identity ext $\mathrm{BM}_t = \mathrm{BM}_t \cap$ ext BM holds. Indeed, if $z^* \in$ ext $\mathrm{BM}_t$ and $z^* = \frac{z_1^*+z_2^*}{2}$ with $z_i^* \in \mathrm{BM}$ then $x_t^* = \frac{Q_2^*z_1^*+Q_2^*z_2^*}{2}$. Since $x_t^* \in$ ext $S^+_{\ell_\infty^*}$ and $Q_2^* z_i^* \in S^+_{\ell_\infty^*}$, we obtain $Q_2^* z_i^* = x_t^*$ and so $z_i^* \in \mathrm{BM}_t$. Thus, $z_1^* = z_2^*$. Finally, we have $z^* \in$ ext BM. Next, the relation $t_1 \neq t_2$ implies $\mathrm{BM}_{t_1} \cap \mathrm{BM}_{t_2} = \emptyset$. Consequently, taking into account the identities card $\widehat{\mathbb{N}} =$ card $\ell_\infty^* = 2^{\mathfrak{c}}$, we infer card (ext BM) $= 2^{\mathfrak{c}}$, and the proof of (a) is completed.

(b) As was mentioned above, $\{Q_1^* x_t^* : t \in \widehat{\mathbb{N}}\}$ is a disjoint system. Therefore, it is enough to establish that $Q_1^* x_t^* \perp$ ext BM for every $t \in \widehat{\mathbb{N}}$. To this end, let $z^* \in$ ext BM, let $t \in \widehat{\mathbb{N}}$, and let $\epsilon > 0$. Pick $k \in \mathbb{N}$ satisfying $\frac{1}{\sqrt{k}} \le \epsilon$. Using the inclusion $D_i^k \in \mathcal{D}(ac_0)$, the equality $\bigcup_{i=1}^k D_i^k = \mathbb{N}$, and the multiplicativity of $z^*$ on $\mathcal{D}(ac_0)$ (see (8) and remarks after the proof of Theorem 3′), we find an index $i_0 \in \{1, \dots, k\}$ such that $z^* D_{i_0}^k = 1$ and, hence, $z^* D_i^k = 0$ for all $i \neq i_0$. Taking into account the preceding lemma, we obtain

$$(z^* \wedge Q_1^* x_t^*)\mathbb{N} \le z^*(\mathbb{N} \setminus D_{i_0}^k) + (Q_1^* x_t^*) D_{i_0}^k = x_t^*(Q_1 \chi_{D_{i_0}^k}) \le \frac{1}{\sqrt{k}} \le \epsilon.$$

Since $\epsilon$ is arbitrary, $z^* \perp Q_1^* x_t^*$, as desired. As a matter of fact, we only used the multiplicativity of the functional $z^* \in S^+_{\ell_\infty^*}$ on $\mathcal{D}(ac_0)$. Therefore, we have, in particular, the relation

$$\{Q_1^* x_t^* : t \in \widehat{\mathbb{N}}\} \perp \overline{\text{ext BM}}^{\sigma(\ell_\infty^*, \ell_\infty)}. \tag{25}$$

(c) Let $D = \{d_1, d_2, \dots\}$ be an infinite subset of $\mathbb{N}$ such that $D \in \mathcal{I}(ac_0)$ and $d_n < d_{n+1}$ for all $n$. Define the operator $T_D$ on the space $\ell_\infty$ via the formula $T_D x = (x_{d_1}, x_{d_2}, \dots)$ for all $x \in \ell_\infty$. Evidently, the range $R(T_D) = \ell_\infty$ and $T_D$ is interval preserving. Whence, $T_D^*$ is a one-to-one lattice homomorphism. Thus, $\{T_D^* Q_1^* x_t^* : t \in \widehat{\mathbb{N}}\}$ is a disjoint system and if $t_1 \neq t_2$ then $T_D^* Q_1^* x_{t_1}^* \neq T_D^* Q_1^* x_{t_2}^*$. Next, as is easy to see, the inclusion $x^* \in \mathrm{BM}$ and the condition $D \in \mathcal{I}(ac_0)$ imply $T_D^* x^* \perp (L_1(\Omega_d) \oplus L_1(\Omega^{\mathrm{BM}}))$. This concludes the proof. $\square$

[1] The author wishes to thank E.M. Semenov and A.S. Usachev for bringing this reference to his attention.

Since $Q_2$ is a lattice homomorphism, the inclusion $Q_2^*(\mathrm{ext}\ S^+_{\ell^*_\infty}) \subseteq \mathrm{ext}\ S^+_{\ell^*_\infty}$ holds. Next, the relation $Q_2^*(\overline{\mathrm{ext\ BM}}^{\sigma(\ell^*_\infty,\ell_\infty)}) \subseteq \mathrm{ext}\ S^+_{\ell^*_\infty}$ holds (compare with (25)). Indeed, we mention first that, in view of Corollary 7(b), the inclusion $R(Q_2) \subseteq \mathcal{D}(ac_0)$ is valid. Let $z^* \in \ell^*_\infty$ be a functional which is multiplicative on $\mathcal{D}(ac_0)$. Then, for every $x, y \in \ell_\infty$, we have

$$(Q_2^* z^*)(xy) = z^*((Q_2 x)(Q_2 y)) = (Q_2^* z^*)x \cdot (Q_2^* z^*)y$$

and, hence, the functional $Q_2^* z^*$ is multiplicative on $\ell_\infty$.

Now we are in a position to give a simple proof of the existence of a functional $x^* \in \mathrm{BM}$ satisfying $x^* \perp \mathrm{ext\ BM}$. Indeed, we claim that if $y^* \in S^+_{L_1(\Omega_c)}$ then $Q_1^* y^* \in S^+_{L_1(\Omega_c^{\mathrm{BM}})}$ and, in particular, $Q_1^* y^* \perp \mathrm{ext\ BM}$. Actually, if $Q_1^* y^* \notin S^+_{L_1(\Omega_c^{\mathrm{BM}})}$ then the inequality $Q_1^* y^* \geq \lambda z^*$ holds for some functional $z^* \in \mathrm{ext\ BM}$ and a number $\lambda > 0$. Therefore, $y^* \geq \lambda Q_2^* z^*$ and, as showed above, $Q_2^* z^* \in L_1(\Omega_d)$, which is impossible.

The proof of Theorem 21(a) suggests to consider the following space. For an arbitrary point $t \in \widehat{\mathbb{N}}$, we define the subspace $L_t$ of $\ell^*_\infty$ by

$$L_t = \{x^* \in N(I - T^*) : Q_2^* x^* \in B_{x_t^*}\},$$

where $B_{x_t^*} = \{\lambda x_t^* : \lambda \in \mathbb{R}\}$ is the band in $\ell^*_\infty$ generated by $x_t^*$. However, $L_t$ is not a Riesz subspace. To see this, pick [11] a functional $z_1^* \in \overline{\mathrm{ext\ BM}}^{\sigma(\ell^*_\infty,\ell_\infty)} \setminus \mathrm{ext\ BM}$. As showed above, $Q_2^* z_1^* = x_{t_1}^*$ for some $t_1 \in \widehat{\mathbb{N}}$. On the other hand, there exists (see the proof of Theorem 21(a)) a functional $z_2^* \in \mathrm{ext\ BM}$ satisfying $Q_2^* z_2^* = x_{t_1}^*$. Let $t_0 \in \widehat{\mathbb{N}}$ and let $z^* \in \mathrm{ext\ BM}$ such that $t_0 \neq t_1$ and $Q_2^* z^* = x_{t_0}^*$. Using Theorem 21(a), we can assume $z^* \perp z_1^* + z_2^*$. Obviously, $z^* + z_1^* - z_2^* \in L_{t_0}$ while

$$Q_2^* |z^* + z_1^* - z_2^*| = x_{t_0}^* + Q_2^* |z_1^* - z_2^*| \notin B_{x_{t_0}^*},$$

as required.

We now turn our attention to estimates of the cardinalities of some subsets of the power set of $\beta\mathbb{N}$.

Let $M$ be a nonempty closed $\varphi$-invariant subset of $\widehat{\mathbb{N}}$. Using Zorn's lemma, it is not difficult to infer the existence of a nonempty closed minimal (by inclusion) $\varphi$-invariant subset $M_0$ of $M$. Obviously, $\varphi(M_0) = M_0$. In particular, for every point $t \in \widehat{\mathbb{N}}$ there exists a nonempty closed minimal $\varphi$-invariant set $\mathcal{M}_t$ satisfying $\varphi(\mathcal{M}_t) = \mathcal{M}_t \subseteq \overline{\mathcal{O}_t}$. According to Theorem 15(c), $\mathcal{M}_t \cap \mathbb{A} \neq \emptyset$, whence we have the inclusion $\mathcal{M}_t \subseteq \mathbb{A}$. Next, for every point $t_1, t_2 \in \widehat{\mathbb{N}}$ either $\mathcal{M}_{t_1} = \mathcal{M}_{t_2}$ or $\mathcal{M}_{t_1} \cap \mathcal{M}_{t_2} = \emptyset$. Put $\mathbb{M} = \bigcup_{t \in \widehat{\mathbb{N}}} \mathcal{M}_t$. If $\varphi(D) \subseteq D$ with $D \subseteq \widehat{\mathbb{N}}$ then $\overline{D} \cap \mathbb{M} \neq \emptyset$. The relation [4] $\overline{\mathbb{M}} \neq \mathbb{A}$ holds and, in particular, there exists a point $t_0 \in \mathbb{A}$ such that $t_0 \notin \mathcal{M}_{t_0}$. Whence $\overline{\mathcal{O}_{t_0}}$ is not a minimal $\varphi$-invariant set. On the other hand, for every point $t \in \mathbb{M}$, we have $\mathcal{M}_t = \overline{\mathcal{O}_t}$. Next, according to Corollary 14, for every point $t \in \widehat{\mathbb{N}}$ there exists a functional $z^*(t) \in \mathrm{ext\ BM}$ satisfying $\mathrm{supp}\ \mu_{z^*(t)} \subseteq \mathcal{M}_t$. Evidently, if $\mathcal{M}_{t_1} \neq \mathcal{M}_{t_2}$ then $z^*(t_1) \perp z^*(t_2)$. The identity [5]

$$\mathrm{card}\ \{\mathcal{M}_t : t \in \widehat{\mathbb{N}}\} = 2^{\mathfrak{c}} \tag{26}$$

holds. In particular, this implies the identity $\mathrm{card}\ (\mathrm{ext\ BM}) = 2^{\mathfrak{c}}$. It should be noted that, as was shown in [12], for every point $t \in \widehat{\mathbb{N}}$ there exist at least two functionals $z_1^*, z_2^* \in \mathrm{ext\ BM}$ such that $\mathrm{supp}\ \mu_{z_i^*} = \mathcal{M}_t$ for $i = 1, 2$. If, in this case, $t \in \mathbb{A}$ then, using Theorem 15(d), we obtain the equality $z_1^* z = z_2^* z$ for all $z \in \mathcal{D}(ac_0)$. Consequently, *the space* $\mathcal{D}(ac_0)$ *does not separate the set* ext BM.

Some more remarks are in order. If for a functional $z^* \in \operatorname{ext} \mathrm{BM}$ and a point $t \in \widehat{\mathbb{N}}$ the identity $\operatorname{supp} \mu_{z^*} = \mathcal{M}_t$ holds then, taking into account Corollary 18, we have $z^* \perp L^{\infty}_{\mathcal{M}_s}$ for every point $s$ satisfying the relation $\mathcal{M}_s \neq \mathcal{M}_t$. Next, since $\overline{\mathbb{M}} \neq \mathbb{A}$, there exists a functional $y^* \in \operatorname{ext} \mathrm{BM}$ such that $\operatorname{supp} \mu_{y^*} \setminus \overline{\mathbb{M}} \neq \emptyset$. Using the identity $\varphi(\mathbb{M}) = \mathbb{M}$, Lemma 13(a), and Corollary 18 once more, we obtain $y^* \perp L^{\infty}_{\mathbb{M}}$. In particular, $y^* \perp L^{\infty}_{\mathcal{M}_t}$ for every point $t \in \widehat{\mathbb{N}}$. Theorem 10 guarantees the existence of a point $t_0 \in \widehat{\mathbb{N}}$ satisfying $\mathcal{M}_{t_0} \subsetneq \operatorname{supp} \mu_{y^*}$. On the other hand, for some functional $x^* \in \operatorname{ext} \mathrm{BM}$, we have $\operatorname{supp} \mu_{x^*} = \mathcal{M}_{t_0} \subseteq \operatorname{supp} \mu_{y^*}$ and $x^* \in L^{\infty}_{\mathcal{M}_{t_0}}$, whence $x^* \perp y^*$.

Consider the space $C(K)$ and an arbitrary nonempty subset $R$ of $C(K)$. We can define an equivalence relation on $K$ by saying that two points $s$ and $t$ in $K$ are equivalent, in symbols $s \approx t$, if $x(s) = x(t)$ for all $x \in R$. Let $\Theta_s$ be the equivalence class of the point $s \in K$, i.e., $\Theta_s = \{t \in K : s \approx t\}$. The collection $\Theta = \{\Theta_s : s \in K\}$ defines a partition of the set $K$ into (pairwise disjoint) closed subsets and is called the *R-partition* of $K$. If $R$ is a linear subspace of $C(K)$, we define the mapping $\Psi : \Theta \to R^*$ via the formula $\Psi(\Theta_s) = \delta_s|_R$, where $\delta_s|_R$ is the restriction of the functional $\delta_s$ onto $R$. As is easy to see, the mapping $\Psi$ is well defined, i.e., if $\Theta_s = \Theta_t$ then $\Psi(\Theta_s) = \Psi(\Theta_t)$, and $\Psi$ is one-to-one.

Now let $R$ be a closed subalgebra of $C(K)$ containing the constant function. In this case, the inclusion $\Psi(\Theta) \subseteq \operatorname{ext} S^+_{R^*}$ holds. On the other hand, every functional $z^* \in \operatorname{ext} S^+_{R^*}$ is a lattice homomorphism. By Lipecki–Luxemburg–Schep theorem [3, p. 99], $z^*$ extends to all of $C(K)$ as a lattice homomorphism. In other words, there exists a point $s \in K$ satisfying $z^* = \delta_s|_R$ and so $z^*x = x(s)$ for all $x \in R$, whence $\Psi$ is a bijection from $\Theta$ onto $\operatorname{ext} S^+_{R^*}$. Consequently, the subalgebra $R$ can be considered as a subalgebra of the algebra $B(\Theta)$ of all bounded functions on $\Theta$. Since the mapping $\xi$ from $R$ into $B(\Theta)$ defined by $(\xi x)(\Theta_s) = x(s)$ is isometric, $\xi(R)$ is closed.

Next, let $y^* \in S^+_{C(K)^*}$ and let the identity $y^*|_R = \delta_s|_R$ hold for some point $s \in K$. By Riesz Representation Theorem [2, p. 497], there exists a unique regular probability Borel measure $\nu$ which defines the functional $y^*$. *The inclusion* $\operatorname{supp} \nu \subseteq \Theta_s$ *is valid*. Indeed, consider a point $t \in K \setminus \Theta_s$. Clearly, $t \in \Theta_t$ and $\Theta_s \neq \Theta_t$. Thus, for some $y \in R^+$, we have $y(t) > 0$ and $y(s) = 0$. Hence, $y^*y = y(s) = 0$ and so $t \notin \operatorname{supp} \nu$.

Now we consider the case of $K = \beta\mathbb{N}$ and

$$R = \{\widehat{z} \in C(\beta\mathbb{N}) : z \in \mathcal{D}(ac_0)\}. \tag{27}$$

As is easy to see if $t \notin \mathbb{A}$ then $\Theta_t = \{t\}$, in particular $\operatorname{card} \{\Theta_t : t \in \beta\mathbb{N} \setminus \mathbb{A}\} = 2^{\mathfrak{c}}$ as every infinite closed subset of $\beta\mathbb{N}$ has the cardinality $2^{\mathfrak{c}}$ (see the remarks at the beginning of Section 3), and if $t \in \mathbb{A}$ then $\overline{\mathcal{O}_t} \subseteq \Theta_t \subseteq \mathbb{A}$ and $\varphi(\Theta_t) = \Theta_t$ as $\widehat{z}$ is constant on $\overline{\mathcal{O}_t}$ for every $z \in \mathcal{D}(ac_0)$ (see Theorem 15(d)). It is not known if for every points $t', t'' \in \Theta_t$ with $t \in \mathbb{A}$ there exists a finite collection of orbits $\mathcal{O}_{t_0}, \mathcal{O}_{t_1}, \ldots, \mathcal{O}_{t_n}$ with $n \in \mathbb{N}$ satisfying $t' \in \overline{\mathcal{O}_{t_0}}, t'' \in \overline{\mathcal{O}_{t_n}}$, and $\overline{\mathcal{O}_{t_i}} \cap \overline{\mathcal{O}_{t_{i+1}}} \neq \emptyset$ for all $i = 0, 1, \ldots, n-1$.

We close this section with some remarks about the cardinality of $\{\Theta_t : t \in \mathbb{A}\}$. For every point $t \in \mathbb{A}$ the inclusion $\mathcal{M}_t \subseteq \Theta_t$ holds and if $\Theta_s \neq \Theta_t$ for some $s \in \mathbb{A}$ then $\mathcal{M}_s \cap \mathcal{M}_t = \emptyset$. Therefore, we have the next estimates

$$\operatorname{card} \{\Theta_t : t \in \mathbb{A}\} \leq \operatorname{card} \{\mathcal{M}_t : t \in \widehat{\mathbb{N}}\} = 2^{\mathfrak{c}}.$$

Next, for every $z_1^*, z_2^* \in \operatorname{ext} \mathrm{BM}$ there exist points $t_1, t_2 \in \mathbb{A}$ satisfying $x^*_{t_i}|_{\mathcal{D}(ac_0)} = z^*_i|_{\mathcal{D}(ac_0)}$ (see the relation (17)). If $z_1^*|_{\mathcal{D}(ac_0)} \neq z_2^*|_{\mathcal{D}(ac_0)}$ then $\Theta_{t_1} \neq \Theta_{t_2}$. However, as was mentioned above, $\mathcal{D}(ac_0)$ does not separate the set ext BM and, in the general case, we will give only the estimate $\operatorname{card} \{\Theta_t : t \in \mathbb{A}\} \geq \mathfrak{c}$ (see Theorem 22).

Before proceeding further, we recall the following well-known fact which will be used in the future. *There exists a collection* $\{A_\alpha : \alpha \in J\}$ *of infinite subsets of* $\mathbb{N}$ *with the cardinality* $\mathfrak{c}$ *which is* **almost disjoint**, i.e., card $(A_{\alpha'} \cap A_{\alpha''}) < \infty$ for all $\alpha' \neq \alpha''$. To prove the existence of such a collection identity the set $\mathbb{N}$ with the set $\mathbb{Q}$ of rational numbers of $\mathbb{R}$ and assign to each real number $\lambda$ a sequence of rational numbers converging to $\lambda$. In this way, we get a collection of subsets of a countable set which has the desired property.

Now we are in position to prove the next estimate of card $\{\Theta_t : t \in \mathbb{A}\}$.

**Theorem 22.** *Let* $\{\Theta_t : t \in \beta\mathbb{N}\}$ *be the* $R$*-partition of* $\beta\mathbb{N}$*, where* $R$ *is defined by* (27)*. The next inequalities* $\mathfrak{c} \leq \text{card}\,\{\Theta_t : t \in \mathbb{A}\} \leq 2^{\mathfrak{c}}$ *hold.*

**Proof.** Let us consider a sequence $\{I_n\}$ of segments of the set $\mathbb{N}$ such that $I_n = [k_n, m_n]$ with $k_n \leq m_n < k_{n+1}$ for all $n$ and $\lim_{n\to\infty}(m_n - k_n) = \infty$. Obviously, $I_i \cap I_j = \emptyset$ for $i \neq j$. Let $\{A_\alpha : \alpha \in J\}$ be an arbitrary almost disjoint collection of subsets of $\mathbb{N}$ with the cardinality $\mathfrak{c}$. Put $D_\alpha = \bigcup_{j\in A_\alpha} I_j$. Evidently, the collection $\{D_\alpha : \alpha \in J\}$ also has the cardinality $\mathfrak{c}$ and is also almost disjoint, whence $\overline{D_{\alpha'}}^{\beta\mathbb{N}} \cap \overline{D_{\alpha''}}^{\beta\mathbb{N}} \cap \widehat{\mathbb{N}} = \emptyset$ for all $\alpha' \neq \alpha''$. Put $x_\alpha = \chi_{D_\alpha} \in \ell_\infty$ and find a point $t_\alpha \in \overline{D_\alpha}^{\beta\mathbb{N}} \cap \widehat{\mathbb{N}}$. In view of Corollary 7(a), $D_\alpha \in \mathcal{D}(ac_0)$ for all $\alpha$ and, hence, $\widehat{x_\alpha} \in R$. Now, using the identities $\widehat{x_\alpha}(s) = 1$ for $s \in \overline{D_\alpha}^{\beta\mathbb{N}}$ and $\widehat{x_\alpha}(s) = 0$ for $s \notin \overline{D_\alpha}^{\beta\mathbb{N}}$, we obtain the relation $t_{\alpha'} \notin \Theta_{t_\alpha}$ with $\alpha' \neq \alpha$ and so $\Theta_{t_{\alpha'}} \neq \Theta_{t_\alpha}$. Consequently, the mapping $D_\alpha \to \Theta_{t_\alpha}$ is one-to-one, and the proof is completed. $\square$

## 5. A bit of a probability

### 5.1. General remarks

Some results above (see, e.g., the inequality (3) or Theorem 3′(h)) suggest an idea about a possibility a glance at some properties of Banach–Mazur limits from the viewpoint of the probability theory. On the other hand, as is well-known (see, e.g., [2, p. 496]), the space $\ell_\infty^*$ is lattice isometric onto the $AL$-space ba$(\mathbb{N}, 2^{\mathbb{N}})$ of all signed finite additive measures (or charges) of bounded variation defined on the power set $2^{\mathbb{N}}$ of $\mathbb{N}$, i.e.,

$$\ell_\infty^* = \text{ba}(\mathbb{N}, 2^{\mathbb{N}}). \tag{28}$$

This isomorphism is defined by the mapping $x^* \to \nu_{x^*}$, where $\nu_{x^*}(A) = x^*\chi_A$ for all subsets $A$ of $\mathbb{N}$. Using our notations, we can simply write $\nu_{x^*}(A) = x^*A$. Thus, elements $\nu_{x^*}$ of the space ba$(\mathbb{N}, 2^{\mathbb{N}})$ will be identified below with functionals $x^*$. For an element $x \in \ell_\infty$, we have

$$x^*x = \int_{\mathbb{N}} x\, d\nu_{x^*} = \int_{\mathbb{N}} x\, dx^*. \tag{29}$$

Below, using the term *measure*, we mean a countable additive signed measure and using the term *finite additive measure*, we mean a signed charge which is not necessarily countable additive (see [2, Chapter 10]). Furthermore, the case of a measure $\mu_{x^*}$ on $\mathcal{B}(\beta\mathbb{N})$ which defines $x^* \in \ell_\infty^*$ via the formula (13) and the case of a finite additive measure $\nu_{x^*}$ on $2^{\mathbb{N}}$ which defines $x^*$ via the formula (29) should differ.

If $x^* \in S^+_{\ell_\infty^*}$ (e.g., $x^* \in$ BM) then $\nu_{x^*}$ is a *probability* finite additive measure, i.e., $\nu_{x^*} \geq 0$ and $\nu_{x^*}(\mathbb{N}) = 1$. As is well-known, $\nu_{x^*}$ *is countable additive iff* $x^* \in \ell_1$; in particular, if $x^* \in$ BM then $\nu_{x^*}$ is not countable additive. Next, if $x^* \in S^+_{\ell_\infty^*}$ then, as is easy to see, $x^* \in$ BM iff $\nu_{x^*}(A) = \nu_{x^*}(A + 1)$ for every subset $A$ of $\mathbb{N}$.

In the probability theory, as a rule, countable additive probability measures were only considered and the case of finite additive measures was ignored. The principal purpose of this section is to take a step in this direction. On the other hand, following the line of research suggested in the preceding sections of the paper, we are only considering finite additive measures on $2^{\mathbb{N}}$. Moreover, the main emphasis will be done on those properties of finite additive probability measures and on notions connected with these measures which can help in a new fashion to take a glance to some properties of Banach–Mazur limits and, in particular, to results obtained below and can be useful for a further research of Banach–Mazur limits.

However, it should be noted that the countable additivity is an important natural assumption both from the viewpoint of some applications of the probability theory and from the viewpoint of a number of theoretical constructions. For instance, without the countable additivity, we be allow seldom to interchange limits and integrals. Moreover, Radon–Nikodym theorem is not valid for finite additive measures (see Section 5.4).

For the convenience of the exposition, this section will be derived by several subsections. In the next subsection, the elementary properties of distribution functions for finite additive measures will be studied. Section 5.3 is devoted to the discussion of possible definitions of a variance and notions connected with its. Radon–Nikodym theorem in the case of finite additive measures is discussed in Section 5.4.

Throughout this section, unless stated otherwise, $x^*$ will stand for a positive functional (a finite additive probability measure) on $\ell_\infty$ with $\|x^*\|_{\ell_\infty^*} = 1$. For information about the finite additive integration of functions, we refer the reader to [6, Sections 3.2, 3.3] (see also [2, Section 11.2]).

### 5.2. Distribution functions

Consider an arbitrary sequence $x \in s$. As in the case of a countable additive probability measure, a *distribution function* of the sequence $x$ (in regard to the functional $x^*$) is the function $F_{x^*,x}$ from $\mathbb{R}$ into $\mathbb{R}$ defined by $F_{x^*,x}(t) = x^*\{n \in \mathbb{N} : x_n \le t\}$. Obviously, $F_{x^*,x}$ is increasing and, hence, is continuous except possibly at countable many points. As can be shown, if $x^* \in \ell_1$ then the set of the discontinuities of $F_{x^*,x}$ coincides with the set $\{x_n : x^*\{n\} > 0\}$ and, in particular, the function $F_{x^*,x}$ is not continuous. On the other hand, the function $F_{x^*,\mathbf{e}}$ is discontinuous for every $x^*$. In Theorem 25 the characterization will be given of functionals $x^*$ such that the function $F_{x^*,x}$ is continuous for some sequence $x \in \ell_\infty$.

Before we will prove two auxiliary results.

**Lemma 23.** *Let $y^* \in \ell_\infty^*$ such that $y^* \perp \ell_1$. Then for every infinite subset $A$ of $\mathbb{N}$ there exists an infinite subset $B$ of $A$ satisfying $y^*B = 0$.*

**Proof.** As is easy to see, we can assume $y^* > 0$ and $y^*A > 0$. Pick an arbitrary almost disjoint collection $\{A_\alpha : \alpha \in J\}$ of subsets of $A$ with the cardinality $\mathfrak{c}$. The condition $y^* \perp \ell_1$ implies the identity $y^*(A_{\alpha'} \cap A_{\alpha''}) = 0$ for $\alpha' \ne \alpha''$. Therefore, it is not difficult to show that the inequality $y^*A_\alpha > 0$ cannot hold for all $\alpha \in J$. $\square$

**Lemma 24.** *Let numbers $\epsilon_1, \dots, \epsilon_k, \epsilon, \lambda$, and $\lambda_0$ satisfy the relations $0 \le \epsilon_i \le \epsilon \le \lambda \le \lambda_0$ and $\sum_{i=1}^k \epsilon_i = \lambda_0 > 0$. Then there exist two disjoint subsets $K_1$ and $K_2$ of $\{1, \dots, k\}$ such that*

$$\sum_{i \in K_1} \epsilon_i \le \lambda, \qquad \sum_{i \in K_2} \epsilon_i \le \lambda_0 - \lambda, \quad \text{and} \quad \sum_{i \in K_1} \epsilon_i + \sum_{i \in K_2} \epsilon_i \ge \lambda_0 - \epsilon \tag{30}$$

*(if $K_j = \emptyset$ for some $j = 1, 2$, we put $\sum_{i \in K_j} \epsilon_i = 0$). If $\epsilon_i > 0$ for all $i = 1, \dots, k$ then* card $(K_1 \cup K_2) = k - 1$.

**Proof.** Clearly, we can suppose $\epsilon_i > 0$ for all $i = 1, \dots, k$. Using a finite induction, we will build the set $K_j$ by steps. On the first step, we put $K_j^1 = \emptyset$. Next, assume that the first $m$ steps have been taken, where $1 \le m \le k$, and disjoint subsets $K_1^m$ and $K_2^m$ have been built satisfying $K_1^m \cup K_2^m \subseteq \{1, \dots, m-1\}$. If $\sum_{i \in K_1^m} \epsilon_i + \epsilon_m \le \lambda$ then we put $K_1^{m+1} = K_1^m \cup \{m\}$ and, otherwise, if $\sum_{i \in K_2^m} \epsilon_i + \epsilon_m \le \lambda_0 - \lambda$ then we put $K_2^{m+1} = K_2^m \cup \{m\}$. If both last inequalities do not hold, we put $K_j^{m+1} = K_j^m$ for $j = 1, 2$. Taking $k$ steps, we either put $K_j = K_j^k$ if card $(K_1^k \cup K_2^k) = k - 1$ or, otherwise, take a last additional step and put $K_j = K_j^{k+1}$. We claim that the sets $K_1$ and $K_2$ obtained by such manner are required. Obviously, the first two inequalities in (30) hold and card $(K_1 \cup K_2) < k$. Next, if card $(K_1 \cup K_2) < k - 1$ then we find two different indexes $i', i'' \in \{1 \dots, k\}$ satisfying $\sum_{i \in K_1} \epsilon_i + \epsilon_{i'} > \lambda$ and $\sum_{i \in K_2} \epsilon_i + \epsilon_{i''} > \lambda_0 - \lambda$, which is impossible. Thus, card $(K_1 \cup K_2) = k - 1$. Pick an index $i_0 \in \{1 \dots, k\}$ such that $i_0 \notin K_1 \cup K_2$. Then $\sum_{i \in K_1} \epsilon_i + \sum_{i \in K_2} \epsilon_i = \lambda_0 - \epsilon_{i_0} \ge \lambda_0 - \epsilon$ and the third inequality in (30) has been established. $\square$

**Theorem 25.** *For a functional $x^* \in S^+_{\ell_\infty^*}$ the following statements are equivalent:*

(a) *For some sequence $x \in \ell_\infty$ the distribution function $F_{x^*,x}$ is continuous;*
(b) *The relation $x^* \perp L_1(\Omega_d)$ holds (see (20));*
(c) *For every $n \in \mathbb{N}$ there exists a partition $N_1, \dots, N_{2^n}$ of $\mathbb{N}$ such that $x^* N_i = \frac{1}{2^n}$ for all $i = 1, \dots, 2^n$;*
(d) *For every $\epsilon > 0$ there exists a partition $N_1, \dots, N_k$ of $\mathbb{N}$ such that $x^* N_i \le \epsilon$ for all $i = 1, \dots, k$;*
(e) *For every $\epsilon > 0$ there exists a cover $N_1, \dots, N_k$ of $\mathbb{N}$ such that $x^* N_i \le \epsilon$ for all $i = 1, \dots, k$;*
(f) *For every subset $A$ of $\mathbb{N}$ there exists a partition $A_1, A_2$ of $A$ such that $x^* A_i = \frac{x^* A}{2}$ for all $i = 1, 2$;*
(g) *For every subset $A$ of $\mathbb{N}$ and a number $\lambda \in [0, x^* A]$ there exists a subset $B$ of $A$ such that $x^* B = \lambda$.*

**Proof.** The implications (g) $\Longrightarrow$ (f) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longleftrightarrow$ (e) are obvious. The implications (b) $\Longleftrightarrow$ (d) are the well-known Sobczyk–Hammer's result (see, e.g., [10]). For the sake of completeness, we include its proof. We also mention that for the case of an arbitrary countable additive measure the equivalence of the statements (c)–(f) is the well-known Saks' results (see, e.g., [2, Section 10.9] and [6, p. 308]) which characterize nonatomic (or diffuse) measures.

(b) $\Longrightarrow$ (e) Fix $\epsilon \in (0, 1)$. For an arbitrary point $t \in \beta\mathbb{N}$ the relation $x^* \perp x_t^*$ holds. Therefore, there exists a subset $A_t$ of $\mathbb{N}$ satisfying $x^* A_t + x_t^*(\mathbb{N} \setminus A_t) \le \epsilon$. Consequently, $x_t^*(\mathbb{N} \setminus A_t) = 0$ and, hence,

$$x^* A_t \le \epsilon \quad \text{and} \quad t \notin \overline{\mathbb{N} \setminus A_t}^{\beta\mathbb{N}}. \tag{31}$$

On the other hand, $\overline{\mathbb{N} \setminus A_t}^{\beta\mathbb{N}} \cup \overline{A_t}^{\beta\mathbb{N}} = \beta\mathbb{N}$ and $\overline{\mathbb{N} \setminus A_t}^{\beta\mathbb{N}} \cap \overline{A_t}^{\beta\mathbb{N}} = \emptyset$ as $\beta\mathbb{N}$ is an extremally disconnected space. Thus, according to the second relation in (31), the point $t$ belongs to the open set $\overline{A_t}^{\beta\mathbb{N}}$. Finally, $\beta\mathbb{N} = \bigcup_{t \in \beta\mathbb{N}} \overline{A_t}^{\beta\mathbb{N}}$. Therefore, $\beta\mathbb{N} = \bigcup_{i=1}^k \overline{A_{t_i}}^{\beta\mathbb{N}}$ for some finite collection of $t_1, \dots, t_k$ and, hence, $\mathbb{N} = \bigcup_{i=1}^k A_{t_i}$. Taking into account the first relation in (31), we have $x^* A_{t_i} \le \epsilon$ for $i = 1, \dots, k$.

(e) $\Longrightarrow$ (b) Proceeding by contradiction, we find a point $t \in \beta\mathbb{N}$ and a number $\gamma > 0$ satisfying the inequality

$$x^* \ge \gamma x_t^*. \tag{32}$$

There exists a cover $N_1, \ldots, N_k$ of $\mathbb{N}$ such that $x^* N_i \le \frac{\gamma}{2}$ for all $i = 1, \ldots, k$. Obviously, $\beta\mathbb{N} = \bigcup_{i=1}^k \overline{N_i}^{\beta\mathbb{N}}$. Therefore, for some index $i_0$, we get $t \in \overline{N_{i_0}}^{\beta\mathbb{N}}$. Whence, using (32), we obtain $\frac{\gamma}{2} \ge x^* N_{i_0} \ge \gamma x_t^* N_{i_0} = \gamma$, which is absurd.

(d) $\Longrightarrow$ (g) Evidently, we can assume $x^* A > 0$ and $\lambda \in (0, x^* A)$. Let $A_1 = A$ and $\lambda_1 = \lambda$. Pick $m_1 \in \mathbb{N}$ satisfying $\frac{1}{m_1} \le \lambda_1$ and a partition $N_{11}, \ldots, N_{1k_1}$ of $\mathbb{N}$ such that $x^* N_{1i} \le \frac{1}{m_1}$ for all $i = 1, \ldots, k_1$. Obviously,

$$x^* A_1 = \sum_{i=1}^{k_1} x^*(A_1 \cap N_{1i}) \quad \text{and} \quad x^*(A_1 \cap N_{1i}) \le \frac{1}{m_1}.$$

Using the preceding lemma, we find two disjoint subsets $K_{11}$ and $K_{12}$ of $\{1, \ldots, k_1\}$ such that for the sets $A_{1j} = \bigcup_{i \in K_{1j}} (A_1 \cap N_{1i})$ with $j = 1, 2$ the inequalities

$$\begin{aligned} &\text{(i) } x^* A_{11} \le \lambda_1, \qquad \text{(ii) } x^* A_{12} \le x^* A_1 - \lambda_1, \quad \text{and} \\ &\text{(iii) } x^* A_{11} + x^* A_{12} \ge x^* A_1 - \frac{1}{m_1} \end{aligned} \tag{33}$$

hold. If at least one of the inequalities (i) and (ii) in (33) is not strict then the proof is finished.

Now assume that both inequalities (i) and (ii) in (33) are strict. Put

$$A_2 = A_1 \setminus (A_{11} \cup A_{12}) \quad \text{and} \quad \lambda_2 = \lambda_1 - x^* A_{11} > 0.$$

Obviously, $x^* A_2 \le \frac{1}{m_1}$. Since

$$\lambda_1 - x^* A_{11} < x^* A_1 - x^* A_{12} - x^* A_{11} = x^* A_2,$$

we have $\lambda_2 \in (0, x^* A_2)$. Pick $m_2 \in \mathbb{N}$ satisfying $m_1 < m_2$ and $\frac{1}{m_2} \le \lambda_2$ and a partition $N_{21}, \ldots, N_{2k_2}$ of $\mathbb{N}$ such that $x^* N_{2i} \le \frac{1}{m_2}$ for all $i = 1, \ldots, k_2$. Obviously,

$$x^* A_2 = \sum_{i=1}^{k_2} x^*(A_2 \cap N_{2i}) \quad \text{and} \quad x^*(A_2 \cap N_{2i}) \le \frac{1}{m_2}.$$

Using the preceding lemma once more, we find two disjoint subsets $K_{21}$ and $K_{22}$ of $\{1, \ldots, k_2\}$ such that for the sets $A'_{2j} = \bigcup_{i \in K_{2j}} (A_2 \cap N_{2i})$ with $j = 1, 2$ the inequalities

$$\begin{aligned} &\text{(i}'\text{) } x^* A'_{21} \le \lambda_2, \qquad \text{(ii}'\text{) } x^* A'_{22} \le x^* A_2 - \lambda_2, \quad \text{and} \\ &\text{(iii}'\text{) } x^* A'_{21} + x^* A'_{22} \ge x^* A_2 - \frac{1}{m_2} \end{aligned}$$

hold. Then, taking into account the definition of $\lambda_2$, we have

$$\begin{aligned} &x^*(A_{11} \cup A'_{21}) \le (\lambda_1 - \lambda_2) + \lambda_2 = \lambda_1, \\ &x^*(A_{12} \cup A'_{22}) \ \le \ x^* A_{12} + x^* A_2 - \lambda_2 \\ &\qquad\qquad\qquad = \ x^* A_{12} + x^* A_1 - x^* A_{11} - x^* A_{12} - \lambda_2 = x^* A_1 - \lambda_1, \end{aligned}$$

and

$$x^* \left( \left( \bigcup_{j=1}^2 A_{1j} \right) \cup \left( \bigcup_{j=1}^2 A'_{2j} \right) \right) = x^* A_1 - x^* A_2 + x^* \left( \bigcup_{j=1}^2 A'_{2j} \right) \ge x^* A_1 - \frac{1}{m_2}.$$

Put $A_{21} = A_{11} \cup A'_{21}$ and $A_{22} = A_{12} \cup A'_{22}$. Finally, we obtain the following relations

$$\text{(i) } x^* A_{21} \le \lambda_1, \qquad \text{(ii) } x^* A_{22} \le x^* A_1 - \lambda_1, \quad \text{and}$$
$$\text{(iii) } x^* A_{21} + x^* A_{22} \ge x^* A_1 - \frac{1}{m_2}.$$

Next, using an easy induction argument, we obtain a strictly increasing sequence $\{m_n\}$ in $\mathbb{N}$ and two sequences $\{A_{n1}\}$ and $\{A_{n2}\}$ in $2^{\mathbb{N}}$ such that $A_{nj} \subseteq A_{n+1,j} \subseteq A$ and $A_{n1} \cap A_{n2} = \emptyset$ for all $n \in \mathbb{N}$ and $j = 1, 2$ and the next inequalities hold

$$\text{(i) } x^* A_{n1} \le \lambda, \qquad \text{(ii) } x^* A_{n2} \le x^* A - \lambda, \quad \text{and}$$
$$\text{(iii) } x^* A_{n1} + x^* A_{n2} \ge x^* A - \frac{1}{m_n}. \tag{34}$$

Put $B_j = \bigcup_{n=1}^{\infty} A_{nj}$ for $j = 1, 2$. Evidently,

$$B_j \subseteq A, \qquad B_1 \cap B_2 = \emptyset, \quad \text{and} \quad x^* A \ge x^* B_1 + x^* B_2 \ge x^* A - \frac{1}{m_n}.$$

Letting $n \to \infty$, we have $x^* A = x^* B_1 + x^* B_2$. Now, using the inequalities (i) and (iii) in (34), we obtain

$$\lambda + x^* A_{n2} \ge x^* A_{n1} + x^* A_{n2} \ge x^* A - \frac{1}{m_n},$$

whence $x^* B_2 \ge x^* A_{n2} \ge x^* A - \frac{1}{m_n} - \lambda$ for all $n$. Letting $n \to \infty$ once more, we have

$$x^* B_2 \ge x^* A - \lambda. \tag{35}$$

Analogously, using the inequalities (ii) and (iii) in (34), we obtain

$$x^* A_{n1} + x^* A - \lambda \ge x^* A_{n1} + x^* A_{n2} \ge x^* A - \frac{1}{m_n}.$$

Therefore, $x^* B_1 \ge x^* A_{n1} \ge \lambda - \frac{1}{m_n}$ and, hence, $x^* B_1 \ge \lambda$. Now, a glance at (35) yields $x^* B_1 = \lambda$ and $x^* B_2 = x^* A - \lambda$, as required.

(a) $\Longrightarrow$ (c) Let the distribution function $f = F_{x^*,x}$ be continuous. Fix $n \in \mathbb{N}$. Since

$$\min_{t \in \mathbb{R}} f(t) = 0 \quad \text{and} \quad \max_{t \in \mathbb{R}} f(t) = 1, \tag{36}$$

for every $i = 0, 1, \ldots, 2^n$ there exists a number $a_i \in \mathbb{R}$ satisfying $f(a_i) = \frac{i}{2^n}$. For $i = 1, \ldots, 2^n$, we define the subsets $N'_i$ of $\mathbb{N}$ by $N'_i = \{n : a_{i-1} < x_n \le a_i\}$. Obviously, we have the identities $x^* N'_i = f(a_i) - f(a_{i-1}) = \frac{1}{2^n}$. Put $N_1 = N'_1 \cup (\mathbb{N} \setminus \bigcup_{i=1}^{2^n} N'_i)$ and $N_i = N'_i$ for $i > 1$, and we are done.

(g) $\Longrightarrow$ (a) We will show that *for an arbitrary increasing (not necessarily continuous) function $f : \mathbb{R} \to \mathbb{R}$ satisfying the relations* (36) *there exists an element $x \in \ell_\infty$ such that $F_{x^*,x} = f$*. To this end, let $a$ and $b$ be two arbitrary numbers with the properties $f(a) = 0$, $f(b) = 1$, and $f$ is continuous at $a$. Clearly, $a < b$. As is easy to see, there exists a sequence $\{t_n\}$ in $(a, b]$ such that $t_1 = b$, $t_i \ne t_j$ for $i \ne j$, and the set $\{t_1, t_2, \ldots\}$ contains the (empty, finite, or countable) collection of all discontinuities of $f$ and is dense in $(a, b]$. The existence of a sequence $x \in \ell_\infty$ satisfying $F_{x^*,x} = f$ can be proved by induction as follows. At the first step, we put $B_{11} = \mathbb{N}$, $m_{11} = k_{11} = 1$, $M_1 = \{m_{11}\}$, $K_1 = \{k_{11}\}$, and $x_{m_{11}} = t_{m_{11}} = b$. We also put $x_0 = a$. Now assume that for some $n$ the partition of $\mathbb{N}$ into infinite subsets $B_{1,2^{n-1}}, \ldots, B_{2^{n-1},2^{n-1}}$ and two collections of subsets $M_1, \dots M_n$ and of subsets $K_1, \dots K_n$ of $\mathbb{N}$ have been constructed satisfying the properties

(1) $M_n = \{m_{1,2^{n-1}}, \dots, m_{2^{n-1},2^{n-1}}\}$ and $m_{j,2^{n-1}} \in B_{j,2^{n-1}}$ for all $j = 1, \dots, 2^{n-1}$,
(2) for indexes $m \in M_n$ elements $x_m$ of $x$ were defined and satisfy the inequalities $a = x_0 < x_{m_{1,2^{n-1}}} < \dots < x_{m_{2^{n-1},2^{n-1}}} = b$,
(3) $x^* B_{j,2^{n-1}} = f(x_{m_{j,2^{n-1}}}) - f(x_{m_{j-1,2^{n-1}}})$ for all $j = 1, \dots, 2^{n-1}$ (we put $m_{j',i'} = 0$ for $j' = 0$ and every $i'$),
(4) $K_n = \{k_{1,2^{n-1}}, \dots, k_{2^{n-1},2^{n-1}}\}$ and $\{t_i : i \in K_n\} = \{x_i : i \in M_n\}$, and if $n \geq 2$ then
(5) $B_{2j-1,2^{n-1}} \cup B_{2j,2^{n-1}} = B_{j,2^{n-2}}$ for all $j = 1, \dots, 2^{n-2}$,
(6) $M_{n-1} \subseteq M_n$ and $K_{n-1} \subseteq K_n$.

For $j = 1, \dots, 2^{n-1}$, we put $m_{2j,2^n} = m_{j,2^{n-1}}$, $k_{2j,2^n} = k_{j,2^{n-1}}$

$$m_{2j-1,2^n} = \min\{k \in B_{j,2^{n-1}} \setminus M_n\},$$
$$k_{2j-1,2^n} = \min\{k \in \mathbb{N} \setminus K_n : t_k \in (x_{m_{j-1,2^{n-1}}}, x_{m_{j,2^{n-1}}})\},$$

and $x_{m_{2j-1,2^n}} = t_{k_{2j-1,2^n}}$. Now we can define

$$M_{n+1} = \{m_{1,2^n}, \dots, m_{2^n,2^n}\} \quad \text{and} \quad K_{n+1} = \{k_{1,2^n}, \dots, k_{2^n,2^n}\}.$$

In view of our condition and Lemma 23, for every $j = 1, \dots, 2^{n-1}$, there exists a partition of $B_{j,2^{n-1}}$ into infinite subsets $B_{2j-1,2^n}$ and $B_{2j,2^n}$ satisfying $m_{2j-1,2^n} \in B_{2j-1,2^n}$, $m_{2j,2^n} \in B_{2j,2^n}$,

$$x^* B_{2j-1,2^n} = f(x_{m_{2j-1,2^n}}) - f(x_{m_{j-1,2^{n-1}}}), \quad \text{and}$$
$$x^* B_{2j,2^n} = f(x_{m_{j,2^{n-1}}}) - f(x_{m_{2j-1,2^n}}).$$

Iterating this procedure, as a result, we define correctly elements $x_k$ of a sequence $x \in \ell_\infty$ for all $k \in \mathbb{N}$. Obviously, if an index $k \in B_{j,2^n}$ with $j = 1, \dots, 2^n$ then $x_{m_{j-1,2^n}} < x_k \leq x_{m_{j,2^n}}$. From the latter, we obtain the identity $\{k : x_k \leq x_{m_{j,2^n}}\} = \bigcup_{i=1}^{j} B_{i,2^n}$. Thus, $F_{x^*,x}(x_{m_{j,2^n}}) = f(x_{m_{j,2^n}})$ and, hence, $F_{x^*,x}(t_n) = f(t_n)$ for all $n$ as $\bigcup_{n=1}^{\infty} K_n = \mathbb{N}$. The set $\{t_1, t_2, \dots\}$ is dense in $(a, b]$ and if a point $t$ does not belong to this set then $f$ is continuous at $t$. Consequently, $F_{x^*,x}(t) = f(t)$ for $t \in (a, b]$. Finally, $F_{x^*,x} = f$ on $\mathbb{R}$ as the *sequence $x$ constructed above satisfies the inequalities $a < x_n \leq b$ for all $n$.*

The proof of theorem is now complete. $\square$

The condition $x \in \ell_\infty$ of part (a) of the preceding theorem is essential. Indeed, if a point $t \in \widehat{\mathbb{N}}$ then, on the one hand, the functional $x_t^* \in L_1(\Omega_d)$, on the other hand, for an arbitrary sequence $x \in s$ satisfying the relation $\lim_{n\to\infty} x_n = +\infty$, we have $F_{x_t^*,x} = 0$.

From the relation $\mathrm{BM} \perp L_1(\Omega_d)$ and the preceding theorem (see the proof of the implication (g) $\Longrightarrow$ (a) and part (g)), we have the following two consequences.

**Corollary 26.** *For every $x^* \in \mathrm{BM}$ and for every increasing function $f : \mathbb{R} \to \mathbb{R}$ satisfying (36) there exists a sequence $x \in \ell_\infty$ such that $F_{x^*,x} = f$.*

**Corollary 27.** *For every $x^* \in \mathrm{BM}$ and for every $\lambda \in [0, 1]$ there exists a subset $A$ of $\mathbb{N}$ such that $x^* A = \lambda$.*

The *atomical part* of $x^* \in \ell_\infty^*$ is the set $\operatorname{Atom} x^* = \{t \in \beta\mathbb{N} : |x^*| \wedge x_t^* > 0\}$. As will be shown in the next theorem, the closure of $\operatorname{Atom} x^*$ in $\beta\mathbb{N}$ can be characterized in terms of distribution functions. For a sequence $x \in \ell_\infty$ and a subset $M$ of $\mathbb{R}$, we put, as usual, $x^{-1}(M) = \{n : x_n \in M\}$.

**Theorem 28.** *Let $x^* \in S^+_{\ell_\infty^*}$. The identities*

$$\overline{\operatorname{Atom} x^*} = \beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a,b]}^{\beta\mathbb{N}} = \beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a,b)}^{\beta\mathbb{N}} \tag{37}$$

*hold, where the union was taken by all sequences $x \in \ell_\infty$ and segments $[a,b]$ such that $F_{x^*,x}$ is continuous on $[a,b]$.*

**Proof.** We check first the inclusion

$$\overline{\operatorname{Atom} x^*} \subseteq \beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a,b]}. \tag{38}$$

If a distribution function $F_{x^*,x}$ is continuous on $[a,b]$ for some sequence $x$ then for every $\epsilon > 0$, we find a partition $B_1, \ldots, B_{k_\epsilon}$ of the set $x^{-1}(a,b]$ satisfying the inequality $x^* B_i \le \epsilon$. Since $\overline{x^{-1}(a,b]} = \bigcup_{i=1}^{k_\epsilon} \overline{B_i}$ and $\epsilon$ is arbitrary, we conclude easily the relation $\operatorname{Atom} x^* \subseteq \beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a,b]}$. The set in the right part of the last inclusion is closed and, hence, (38) has been established.

The proof will be completed if we can verify that $\beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a,b)} \subseteq \overline{\operatorname{Atom} x^*}$. To see this, consider a point $t \notin \overline{\operatorname{Atom} x^*}$. If $t \in \mathbb{N}$ then we consider $x = \mathbf{e}_n$ and an interval $(a,b)$ satisfying $1 \in (a,b) \subseteq [0,+\infty)$. Obviously, $F_{x^*,x}$ is continuous on $[a,b]$ and $t \in x^{-1}(a,b)$. Now let $t \in \widehat{\mathbb{N}}$. There exists a subset $D$ of $\mathbb{N}$ such that $t \in \overline{D}$ and $\overline{D} \cap \overline{\operatorname{Atom} x^*} = \emptyset$. Clearly, $\operatorname{card} D = \infty$. Let $D = \{d_1, d_2, \ldots\}$ with $d_i < d_{i+1}$ for all $i \in \mathbb{N}$. Define the functional $y^* \in \ell_\infty^*$ via the formula $y^* z = x^* z_D$ for all $z \in \ell_\infty$, where the sequence $z_D \in \ell_\infty$ defined by $(z_D)_{d_n} = z_n$ for all $n$ and $(z_D)_n = 0$ for all $n \notin D$. The relation $y^* \perp L_1(\Omega_d)$ holds. To see this, fix $\epsilon > 0$. For an arbitrary point $s \in \overline{D}$ there exists a subset $D_s$ of $\mathbb{N}$ satisfying $s \in \overline{D_s}$ and $x^* D_s \le \epsilon$. Therefore, $\overline{D} \subseteq \bigcup_{i=1}^k \overline{D_{s_i}} = \overline{\bigcup_{i=1}^k D_{s_i}}$ for some points $s_1, \ldots, s_k$. Whence, we obtain $D \subseteq \bigcup_{i=1}^k D_{s_i}$. For $i = 1, \ldots, k$, we define the sets $N_i = \{n : d_n \in D_{s_i}\}$. Evidently, $\bigcup_{i=1}^k N_i = \mathbb{N}$ and $y^* N_i \le x^* D_{s_i} \le \epsilon$. In view of part (e) of the preceding theorem, $y^* \perp L_1(\Omega_d)$. Next, according to part (a) of this theorem (see also the proof of the implication (g) $\Longrightarrow$ (a)), there exists a sequence $y \in \ell_\infty$ satisfying $F_{y^*,y}(u) = u$ for all $u \in [0,1]$ and $0 < y_n \le 1$ for all $n$. Fix $\delta > 0$ and pick a sequence $x \in \ell_\infty$ such that $x_{d_n} = y_n$ for all $n$ and $x_n > 1 + \delta$ for all $n \notin D$. Then for every number $u \le 1 + \delta$, we have the equalities

$$\begin{aligned} F_{y^*,y}(u) &= y^*\{n : y_n \le u\} = x^*\{d_n : y_n \le u\} \\ &= x^*\{d_n : x_{d_n} \le u\} = x^*\{n : x_n \le u\} = F_{x^*,x}(u). \end{aligned}$$

In particular, $F_{x^*,x}$ is continuous on $[0, 1+\delta]$. Using the identity $x^{-1}(0, 1+\delta) = D$, we obtain $t \in \overline{D} = \overline{x^{-1}(0, 1+\delta)}$, as desired. □

The equalities (37) can be considered as making more precise of the implication (a) $\Longrightarrow$ (b) of Theorem 25. The preceding theorem remains valid under the assumption $x \in s$.

As follows from (37), the inclusion $\beta\mathbb{N} \setminus \bigcup \overline{x^{-1}[a,b)} \subseteq \overline{\operatorname{Atom} x^*}$ holds. However, this inclusion can be proper. To see this, it suffices to observe that $F_{x^*,0}$ is continuous on $[0,\lambda]$ for every $\lambda > 0$ and $x^{-1}[0,\lambda) = \mathbb{N}$ if $x = 0$. Next, considering a sequence $x \in c_0$ with either $x_n > 0$ or $x_n < 0$ for all $n$ and a functional $x^* \in \ell_1^{\mathrm{d}}$, it is easy to see that in Theorem 28 the condition about the continuity of $F_{x^*,x}$ on $[a, b]$ cannot be replaced by the condition about the continuity on one of the sets $(a, b)$, $(a, b]$, or $[a, b)$. On the other hand, if $x^* \in \ell_1$ then the identity $\overline{\text{Atom } x^*} = \beta\mathbb{N} \setminus \bigcup \overline{x^{-1}(a, b]}$ holds, where the union was taken by all sequences $x \in \ell_\infty$ and segments $[a, b]$ such that $F_{x^*,x}$ is continuous on $(a, b]$. Indeed, let $n \in \mathbb{N}$ and let $x^*\{n\} > 0$, i.e., $n \in \text{Atom } x^*$. If $n \in x^{-1}(a, b]$ for $x \in \ell_\infty$ and $F_{x^*,x}$ is continuous on $(a, b]$ then $x^*\{n\} = 0$, a contradiction.

By analogy with the probability theory, for sequence $x \in \ell_\infty$ the value $x^*x$ can be considered as the *expectation* of $x$ (with the respect of the functional $x^*$). On the other hand, an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ generates the *superposition operator* on the space $s$ (see, e.g., [1]) defined by $\mathbf{f}x = (f(x_1), f(x_2), \ldots)$ for all $x \in s$. As is well-known, if $x^* \in \ell_1$ and $f$ is a Borel function then the relations $F_{x^*,y} = F_{x^*,z}$ and $\mathbf{f}y, \mathbf{f}z \in \ell_\infty$ with $y, z \in s$ imply $x^*\mathbf{f}y = x^*\mathbf{f}z$. An analogous result holds if $x^* \in \ell_\infty^*$. The details are included in the next proposition.

**Proposition 29.** *Let $x^* \in S^+_{\ell_\infty^*}$ and let a function $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then the identity $F_{x^*,y} = F_{x^*,z}$, where the elements $y, z \in \ell_\infty$, implies $x^*\mathbf{f}y = x^*\mathbf{f}z$.*

**Proof.** Fix $\epsilon > 0$ and a number $M$ satisfying $M > \max\{\|y\|, \|z\|\}$. Find a collection of scalars $\lambda_0, \lambda_1, \ldots, \lambda_m$ with $m \in \mathbb{N}$ such that $-M = \lambda_0 < \lambda_1 < \cdots < \lambda_m = M$ and $\lambda_i - \lambda_{i-1} < \epsilon$ for $i = 1 \ldots, m$. Define the sets $A_i = \{n : \lambda_{i-1} < y_n \le \lambda_i\}$ and $B_i = \{n : \lambda_{i-1} < z_n \le \lambda_i\}$. Obviously, $x^*A_i = x^*B_i$ and each of two collections $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ is a partition of $\mathbb{N}$. For every $k \in \mathbb{N}$, we have the relations

$$\Big|y^k - \sum_{i=1}^m \lambda_i^k \chi_{A_i}\Big| = \Big|y^k - \Big(\sum_{i=1}^m \lambda_i \chi_{A_i}\Big)^k\Big| \le kM^{k-1}\Big|y - \sum_{i=1}^m \lambda_i \chi_{A_i}\Big| \le \epsilon k M^{k-1}.$$

Thus, $|x^*y^k - \sum_{i=1}^m \lambda_i^k x^*A_i| \le \epsilon k M^{k-1}$. Analogously, $|x^*z^k - \sum_{i=1}^m \lambda_i^k x^*B_i| \le \epsilon k M^{k-1}$. Since $\epsilon$ is arbitrary, we infer $x^*y^k = x^*z^k$. Consequently, the equality $x^*\mathbf{f}y = x^*\mathbf{f}z$ holds if $f$ is a polynomial and, hence, in view of Weierstrass theorem, if $f$ is a continuous function. □

The preceding proposition does not hold for an arbitrary Borel function $f$. Indeed, we define the function $f$ by $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t \ge 0$. If $x^* \perp \ell_1$ and $z \in c_0$ with $z_n < 0$ for each $n$ then $F_{x^*,0} = F_{x^*,z}$ while $1 = x^*\mathbf{f}0 \ne x^*\mathbf{f}z = 0$.

For an arbitrary sequence $x \in \ell_\infty$ the *characteristic function* of $x$ (with respect of $x^* \in \ell_\infty^*$) is the function $f_{x^*,x}$ from $\mathbb{R}$ into $\mathbb{C}$ defined by

$$f_{x^*,x}(t) = x^*e^{itx} := x^*(\cos(tx_1), \cos(tx_2), \ldots) + ix^*(\sin(tx_1), \sin(tx_2), \ldots).$$

**Corollary 30.** *Let $x^* \in S^+_{\ell_\infty^*}$ and let $y, z \in \ell_\infty$. If $F_{x^*,y} = F_{x^*,z}$ then $f_{x^*,y} = f_{x^*,z}$.*

As is well-known from the probability theory, if a functional $x^* \in \ell_1$ then the identity $f_{x^*,y} = f_{x^*,z}$ implies $F_{x^*,y} = F_{x^*,z}$. In the general case of $x^* \in \ell_\infty^*$ this assertion is not valid. Indeed, let $x^* \perp \ell_1$ and let $y, z \in c_0$ such that $y_n < 0$ and $z_n > 0$ for all $n$. Then $f_{x^*,y}(t) = f_{x^*,z}(t) = 1$ for all $t$ while $F_{x^*,y} \ne F_{x^*,z}$.

### 5.3. Two definitions of a variance

In the probability theory the next "standard" definition of a variance is used. Namely, the *variance* of an element $x \in \ell_\infty$ is the value $\mathbb{D}_{x^*}x = x^*((x - (x^*x)\mathbf{e})^2)$. As in the case of $x^* \in \ell_1$, the most of elementary properties of the variance remains valid in this general case and it can be checked without difficulty.

Now let $z^* \in \operatorname{ext} \mathrm{BM}$. We have the next statement: *the identity* $\mathbb{D}_{z^*} z = 0$ *holds iff* $z \in \mathcal{D}_{z^*}$. Indeed, if $\mathbb{D}_{z^*} z = 0$ then, using the Cauchy–Schwarz inequality $(x^*(vw))^2 \le x^*(v^2) \cdot x^*(w^2)$ which is valid for arbitrary elements $v, w \in \ell_\infty$, we obtain $z^*|z - (z^*z)\mathbf{e}| \le (\mathbb{D}_{z^*} z)^{\frac{1}{2}} = 0$. In view of Theorem 3(d), $z \in \mathcal{D}_{z^*}$. For the converse, let $z \in \mathcal{D}_{z^*}$. Taking into account Corollary 4, we have $\mathbb{D}_{z^*} z = (z^*(z - (z^*z)\mathbf{e}))^2 = 0$.

Next, the *covariance* between two sequences $y, z \in \ell_\infty$ is the value

$$\operatorname{cov}_{x^*}(y, z) = x^*((y - (x^*y)\mathbf{e})(z - (x^*z)\mathbf{e})).$$

As is easy to see, the identity $\mathbb{D}_{x^*}(y + z) = \mathbb{D}_{x^*} y + 2\operatorname{cov}_{x^*}(y, z) + \mathbb{D}_{x^*} z$ holds. It follows from the remarks above that if $z^* \in \operatorname{ext} \mathrm{BM}$ and $y, z \in \mathcal{D}_{z^*}$ then $\operatorname{cov}_{z^*}(y, z) = 0$.

We have the following variant of the Law of large numbers.

**Proposition 31.** *Let* $x^* \in S^+_{\ell^*_\infty}$ *and let* $\{x_n\}$ *be a sequence in* $\ell_\infty$ *such that* $\operatorname{cov}_{x^*}(x_i, x_j) = 0$ *for* $i \neq j$ *and the sequence* $\{\mathbb{D}_{x^*} x_n\}$ *is bounded. Then for every* $\epsilon > 0$ *the next relation holds*

$$\lim_{k \to \infty} x^* \left\{ n : \left| \frac{(x_1)_n + \cdots + (x_k)_n}{k} - \frac{x^*x_1 + \cdots + x^*x_k}{k} \right| \ge \epsilon \right\} = 0. \tag{39}$$

**Proof.** Put $d = \sup_n \mathbb{D}_{x^*} x_n$. Using Lemma 1, we have

$$\begin{aligned} x^* \left\{ n : \left| \frac{1}{k} \sum_{i=1}^{k} ((x_i)_n - x^*x_i) \right| \ge \epsilon \right\} &\le \frac{1}{\epsilon^2 k^2} x^* \left( \left( \sum_{i=1}^{k} (x_i - (x^*x_i)\mathbf{e}) \right)^2 \right) \\ &= \frac{1}{\epsilon^2 k^2} \sum_{i=1}^{k} \mathbb{D}_{x^*} x_i \le \frac{d}{\epsilon^2 k} \to 0 \end{aligned}$$

as $k \to \infty$. $\square$

In usual form of the Law of large numbers, the notion of independence is often used. Recall that if two random variables $\xi$ and $\eta$ are independent then for them expectations, we have $E(\xi\eta) = E\xi \cdot E\eta$ and, hence, $E((\xi - E\xi)(\eta - E\eta)) = 0$. This is just the condition what we required in the preceding proposition for the sequence $\{x_n\}$. As was mentioned above, this condition holds if $x^* \in \operatorname{ext} \mathrm{BM}$ and $x_n \in \mathcal{D}_{x^*}$ and follows from the multiplicativity of $x^*$ on $\mathcal{D}_{x^*}$. Thus, in this case the property of the independence can be replaced by the property of the multiplicativity.

Nevertheless, some results of the preceding sections (see, e.g., Theorem 3′(d)(h)) and some remarks done in this subsection suggest another possible definition of a variance which can be more suitable from the viewpoint of the study of Banach–Mazur limits. Namely, the *variance* of an element $x \in \ell_\infty$ is the value $\mathbb{D}_{x^*} x = x^*|x - (x^*x)\mathbf{e}|$. Again, many elementary properties of the "standard" variance remain valid in this case. Below, to avoid ambiguity, if we will write $\mathbb{D}_{x^*} x$ and will say a "variance" then we will mean this second definition.

Let $z \in \ell_\infty$. The next two statements follow from the identity (2) and Theorem 3:

(a) If $z^* \in \operatorname{ext} \mathrm{BM}$ then $z \in \mathcal{D}_{z^*}$ iff $\mathbb{D}_{z^*} z = 0$;
(b) $z \in \mathcal{D}(ac_0)$ iff $\mathbb{D}_{z^*} z = 0$ for all $z^* \in \operatorname{ext} \mathrm{BM}$.

**Proposition 31′.** *Let* $x^* \in S^+_{\ell^*_\infty}$ *and let* $\{x_n\}$ *be a sequence in the space* $\ell_\infty$ *such that* $\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mathbb{D}_{x^*} x_i = 0$ *(e.g.,* $x^* \in \operatorname{ext} \mathrm{BM}$ *and* $x_n \in \mathcal{D}_{x^*}$ *for sufficiently large n). Then for every* $\epsilon > 0$ *the relation (39) holds.*

**Proof.** Using Lemma 1, we have

$$x^*\Big\{n : \Big|\frac{1}{k}\sum_{i=1}^{k}((x_i)_n - x^*x_i)\Big| \geq \epsilon\Big\} \leq \frac{1}{\epsilon k}x^*\Big|\sum_{i=1}^{k}x_i - x^*\Big(\sum_{i=1}^{k}x_i\Big)\mathbf{e}\Big|$$
$$\leq \frac{1}{\epsilon k}\sum_{i=1}^{k}x^*|x_i - (x^*x_i)\mathbf{e}| \to 0$$

as $k \to \infty$. $\square$

As the next proposition shows, minimum points for the function $\Phi_{x^*,x}(t) = x^*|x - t\mathbf{e}|$ with $t \in \mathbb{R}$ can be characterized in terms of a distribution function. An analogous result for countable additive probability measures was mentioned in [14, p. 44, Exercise 5].

**Proposition 32.** *Let $x^* \in S^+_{\ell^*_\infty}$ and let $x \in \ell_\infty$. If $x^*\{n : x_n < t_0\} \leq \frac{1}{2} \leq x^*\{n : x_n \leq t_0\}$ for some number $t_0$ then $x^*|x - t\mathbf{e}| \geq x^*|x - t_0\mathbf{e}|$ for all $t \in \mathbb{R}$ and, in particular, $\mathbb{D}_{x^*}x \geq x^*|x - t_0\mathbf{e}|$.*

**Proof.** We will show first that if $F_{x^*,y}(0) \geq \frac{1}{2}$ for some $y \in \ell_\infty$ then $x^*|y| \leq x^*|y - t\mathbf{e}|$ for all $t \geq 0$. Indeed,

$$\begin{aligned} x^*(y^+ \wedge (t\mathbf{e})) &= x^*P_{\{n:y_n>0\}}(y^+ \wedge (t\mathbf{e})) \leq tx^*\{n : y_n > 0\} \\ &= t(1 - F_{x^*,y}(0)) \leq t\left(1 - \frac{1}{2}\right) = \frac{t}{2}. \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned} x^*((y - t\mathbf{e})^- \wedge (t\mathbf{e})) &= x^*P_{\{n:y_n<t\}}((y - t\mathbf{e})^- \wedge (t\mathbf{e})) \\ &\geq x^*P_{\{n:y_n\leq 0\}}((y - t\mathbf{e})^- \wedge (t\mathbf{e})) = x^*P_{\{n:y_n\leq 0\}}t\mathbf{e} \geq \frac{t}{2}. \end{aligned}$$

Whence, using (40), we have $x^*(y^+ \wedge (t\mathbf{e})) \leq x^*((y - t\mathbf{e})^- \wedge (t\mathbf{e}))$. Now taking into account the last inequality and the identities

$$(y - t\mathbf{e})^- \wedge (t\mathbf{e}) + y^- = (y - t\mathbf{e})^- \quad \text{and} \quad y^+ - y^+ \wedge (t\mathbf{e}) = (y - t\mathbf{e})^+,$$

we obtain

$$\begin{aligned} x^*|y| &= x^*(y^+ \wedge (t\mathbf{e}) + y^- + (y^+ - y^+ \wedge (t\mathbf{e}))) \\ &\leq x^*(((y - t\mathbf{e})^- \wedge (t\mathbf{e}) + y^-) + (y^+ - y^+ \wedge (t\mathbf{e}))) \\ &= x^*((y - t\mathbf{e})^- + (y - t\mathbf{e})^+) = x^*|y - t\mathbf{e}|. \end{aligned}$$

Now let the inequality $x^*\{n : y_n < 0\} \leq \frac{1}{2}$ hold for some $y \in \ell_\infty$. Then $F_{x^*,-y}(0) \geq \frac{1}{2}$. Therefore, as showed above, $x^*|y| \leq x^*|(-y) - (-t)\mathbf{e}| = x^*|y - t\mathbf{e}|$ for all $t \leq 0$. Thus, the required assertion has been proved in the case of $t_0 = 0$.

In the general case, using the inequalities $x^*\{n : x_n - t_0 < 0\} \leq \frac{1}{2} \leq F_{x^*,x-t_0\mathbf{e}}(0)$, we have $x^*|x - t_0\mathbf{e}| \leq x^*|x - (t_0 + t)\mathbf{e}|$ for every $t \in \mathbb{R}$, as desired. $\square$

The converse to the statement of the preceding proposition is not valid. Actually, if $x^* \perp \ell_1$ and $x \in c_0$ then $\Phi_{x^*,x}(t) \geq \Phi_{x^*,x}(0) = 0$. But, if we also assume $x_n < 0$ for all $n$ then $x^*\{n : x_n < 0\} = 1$.

On the other hand, as follows from Theorem 3(d), if $z^* \in \operatorname{ext} \mathrm{BM}$ and $z \in \mathcal{D}_{z^*}$ then $\min_{t\in\mathbb{R}} \Phi_{z^*,z}(t) = 0$ and $\Phi_{z^*,z}$ attains its minimum at the point $z^*z$. In the next example this case will be considered in detail.

**Example 33.** Let $z^* \in \operatorname{ext} \mathrm{BM}$ and let $z \in \ell_\infty$. Then $z \in \mathcal{D}_{z^*}$ *iff*

$$F_{z^*,z}(t) = 0 \quad \text{for} \quad t < z^*z \quad \text{and} \quad F_{z^*,z}(t) = 1 \quad \text{for} \quad t > z^*z. \tag{41}$$

We shall prove first the necessity. For an arbitrary number $t < z^*z$, we find $\epsilon > 0$ satisfying $t + \epsilon \leq z^*z$. Using Theorem 3(h), we obtain

$$F_{z^*,z}(t) = z^*\{n : z_n \leq t\} \leq z^*\{n : z_n \leq z^*z - \epsilon\} \leq z^*\{n : |z_n - z^*z| \geq \epsilon\} = 0.$$

The second equality in (41) can be checked in a similar manner. For the converse, for every $\epsilon > 0$, we have the identities $z^*\{n : z_n \leq z^*z - \epsilon\} = 0$ and $z^*\{n : z_n < z^*z + \epsilon\} = 1$. Therefore, $z^*\{n : |z_n - z^*z| \geq \epsilon\} = 0$ and so $z \in \mathcal{D}_{z^*}$.

Nevertheless, even when $z \in \mathcal{D}_{z^*}$, every number in the segment $[0, 1]$ can be a value of $F_{z^*,z}$ at the point $z^*z$. Indeed, using Corollary 27, for an arbitrary scalar $\lambda \in [0, 1]$, we find a subset $A$ of $\mathbb{N}$ satisfying $z^*A = \lambda$ and consider a sequence $z \in \ell_\infty$ which converges to $\lambda$ and satisfies the relations $z_n \leq \lambda$ for all $n \in A$ and $z_n > \lambda$ for all $n \notin A$. Obviously, $z \in \mathcal{D}_{z^*}$ and $F_{z^*,z}(z^*z) = \lambda$. $\square$

### 5.4. *Radon–Nikodym theorem*

Let $x^*, y^* \in \ell_\infty^*$ be two finite additive measures on $\mathbb{N}$. The measure $x^*$ is said to be *absolutely continuous* (see, e.g., [2, Section 10.12]) with respect to the measure $y^*$, written $x^* \ll y^*$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every subset $A$ of $\mathbb{N}$ the inequality $|y^*|A < \delta$ implies $|x^*|A < \epsilon$. As is well-known (see [2, p. 401]), *the relation $x^* \ll y^*$ holds iff $x^* \in B_{y^*}$.* Next, *Radon–Nikodym theorem* (see, e.g., [2, Section 13.6]) asserts, in particular, the following: *if $x^*, y^* \in \ell_1 \subseteq \ell_\infty^*$ and $x^* \ll y^*$ (i.e., in other words, $y^*\mathbf{e}_n = 0$ for some $n$ implies $x^*\mathbf{e}_n = 0$) then there exists a unique $y^*$-integrable sequence $w \in s$ satisfying the identity*

$$x^*A = \int_A w \, dy^* \tag{42}$$

*for every subset $A$ of* $\mathbb{N}$; in this case, the sequence $w$ can be defined by $w_n = \frac{x^*\mathbf{e}_n}{y^*\mathbf{e}_n}$ if $y^*\mathbf{e}_n \neq 0$ and $w_n = 0$ otherwise. If $w \in \ell_\infty$ then (42) is equivalent to the identity

$$x^*x = y^*(wx) \tag{43}$$

for all $x \in \ell_\infty$, i.e., $x^* = y_w^*$ (see (5)).

As is well-known, Radon–Nikodym theorem is the most prominent result for the construction of conditional expectations (see, e.g., [14, Section 2.7]). Therefore, in the study of finite additive probability measures the question about the validity of this result in the general case arises naturally. Unfortunately, as Example 36 shows, *Radon–Nikodym theorem is not valid for finite additive measures.*

The next lemma will be needed later.

**Lemma 34.** *Let $x^*, y^* \in \ell_\infty^*$ be two functionals such that $0 \leq x^* \leq y^*$ and let the identity (42) hold for some $w \in s$ and all subsets $A$ of $\mathbb{N}$. Then there exists a sequence $w_0 \in \ell_\infty$ satisfying $0 \leq w_0 \leq \mathbf{e}$ and $\int_A w \, dy^* = \int_A w_0 \, dy^*$ for all subsets $A$ of $\mathbb{N}$.*

**Proof.** For an arbitrary subset $A$ of $\mathbb{N}$, we have

$$0 \leq \int_A w^- \, dy^* = \int_{A \cap \{n : w_n < 0\}} w^- \, dy^* \leq \int_{A \cap \{n : w_n < 0\}} w^+ \, dy^* = 0$$

and so $\int_A w^- \, dy^* = 0$. Thus, $\int_A w \, dy^* = \int_A w^+ \, dy^*$ and, hence, we can assume $w \geq 0$ (we only used the inequalities $y^* \geq 0$ and $\int_B w \, dy^* \geq 0$ for all subsets $B$ of $\mathbb{N}$). Next, using (42), we obtain $\int_A w \, dy^* \leq y^* A = \int_A \mathbf{e} \, dy^*$. Therefore, $\int_A (\mathbf{e} - w) \, dy^* \geq 0$. As showed above, the relation $\int_A (\mathbf{e} - w) \, dy^* = \int_A (\mathbf{e} - w)^+ \, dy^*$ holds and so $\int_A w \, dy^* = \int_A (\mathbf{e} - (\mathbf{e} - w)^+) \, dy^*$. It remains to observe the validity of the inequalities $0 \leq w_0 \leq \mathbf{e}$ with $w_0 = \mathbf{e} - (\mathbf{e} - w)^+$. $\square$

First of all, we mention the conditions under which Radon–Nikodym theorem is valid.

**Proposition 35.** *Let $x^*, y^* \in \ell_\infty^*$ and let $x^* \ll y^*$. We have the next statements:*

(a) *If $y^* = \sum_{i=1}^k \alpha_i x_{t_i}^*$ with $t_i \in \beta\mathbb{N}$ and $\alpha_i \in \mathbb{R}$ then the identity (43) holds for some $w \in \ell_\infty$ and for all $x \in \ell_\infty$;*

(b) *If there exist a sequence $\{t_n\}$ in $\beta\mathbb{N}$ and a sequence $\{x_n\}$ in $\ell_\infty$ satisfying $y^* = \sum_{i=1}^\infty \alpha_i x_{t_i}^*$ with $\alpha \in \ell_1$, $x_{t_n}^* x_n \neq 0$ and $x_{t_n}^* \left( \bigcup_{j \neq n} \operatorname{supp} x_j \right) = 0$ for all $n$, and for every $k \in \mathbb{N}$*

$$\sup \left\{ m \in \mathbb{N} : \ k \in \bigcap_{j=1}^m \operatorname{supp} x_{n_j} \text{ for some } n_1, \ldots, n_m, \ n_i \neq n_j \text{ for } i \neq j \right\} < \infty \tag{44}$$

*then the identity (42) holds for some $w \in s$ and for every subset $A$ of $\mathbb{N}$.*

**Proof.** (a)We can suppose $t_i \neq t_j$ for $i \neq j$. Since $B_{y^*} \subseteq L_{\{t_1, \ldots, t_k\}}$ and $x^* \in B_{y^*}$, we have $x^* = \sum_{i=1}^k \alpha_i' x_{t_i}^*$ with $\alpha_i' \in \mathbb{R}$. Next, let $A_1, \ldots, A_k$ be a partition of $\mathbb{N}$ such that $t_i \in \overline{A_i}^{\beta\mathbb{N}}$ for $i = 1, \ldots, k$. Assuming $\alpha_i \neq 0$, we put $w = \sum_{i=1}^k \frac{\alpha_i'}{\alpha_i} \chi_{A_i}$. Since $x_{t_i}^* w = \frac{\alpha_i'}{\alpha_i}$, we have

$$y^*(wx) = \sum_{i=1}^k \alpha_i x_{t_i}^*(wx) = \sum_{i=1}^k \alpha_i' x_{t_i}^* x = x^* x,$$

as required.

(b) For some sequence $\alpha' \in \ell_1$ the identity $x^* = \sum_{i=1}^\infty \alpha_i' x_{t_i}^*$ holds. We can suppose $x_{t_n}^* x_n = 1$ for all $n$. Moreover, using part (a), we can assume $\alpha_i \neq 0$ for all $i$. Put $\lambda_i = \frac{\alpha_i'}{\alpha_i}$ and define the sequence $w \in s$ (not necessarily bounded) by $w_k = \sum_{i=1}^\infty \lambda_i (x_i)_k$. In view of (44), $w$ is well defined. We claim that $w$ satisfies the identity (42). To this end, consider the sequence $\{w^{(n)}\}$ in $\ell_\infty$ defined by $w^{(n)} = \sum_{i=1}^n \lambda_i x_i$. The sequence $\{w^{(n)}\}$ converges in measure $y^*$ to $w$, i.e.,

$$w^{(n)} \xrightarrow{y^*} w. \tag{45}$$

Indeed, for every $\epsilon > 0$, we have

$$|y^*|\{k : |w_k^{(n)} - w_k| \geq \epsilon\} \leq |y^*| \left( \bigcup_{j=n+1}^\infty \operatorname{supp} x_j \right) = \sum_{i=1}^\infty |\alpha_i| x_{t_i}^* \left( \bigcup_{j=n+1}^\infty \operatorname{supp} x_j \right)$$
$$= \sum_{i=n+1}^\infty |\alpha_i| \to 0$$

as $n \to \infty$. Next, we will show the relation

$$\lim_{|y^*| E \to 0} \int_E |w^{(n)}| \, d|y^*| = 0 \tag{46}$$

uniformly in $n$. To see this, fix $\epsilon > 0$ and choose an index $n_0$ satisfying $\sum_{i=n_0+1}^\infty |\alpha_i'| \leq \epsilon$. Next, pick $\delta > 0$ such that $\delta < \min\{|\alpha_1|, \ldots, |\alpha_{n_0}|\}$. Therefore, for an arbitrary subset $E$ of $\mathbb{N}$ the inequality $|y^*|E \le \delta$ implies the relation $\{t_1,\ldots,t_{n_0}\}\cap \overline{E}^{\beta\mathbb{N}} = \emptyset$. Whence, for all $n > n_0$, we get

$$\int_E |w^{(n)}|\,d|y^*| \le \int_E \sum_{i=1}^{n} |\lambda_i|\,|x_i|\,d|y^*| = \sum_{i=1}^{n} |\lambda_i|\,|y^*|(P_E|x_i|) \le \sum_{i=n_0+1}^{n} |\alpha_i'| \le \epsilon,$$

and (46) has been established. Using (45) and (46), we infer [6, p. 122] that $w$ is $y^*$-integrable and $\int_{\mathbb{N}} |w - w^{(n)}|\,d|y^*| \to 0$ as $n \to \infty$. Finally, for an arbitrary subset $A$ of $\mathbb{N}$, we have

$$\begin{aligned}\int_A w\,dy^* &= \lim_{n\to\infty}\int_A w^{(n)}\,dy^*\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\int_A \lambda_i x_i\,dy^* = \lim_{n\to\infty}\sum_{\substack{i\in\{1,\ldots,n\}\\ t_i\in\overline{A}^{\beta\mathbb{N}}}} \alpha_i' = \lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i' x_{t_i}^* A = x^*A,\end{aligned}$$

and the proof is completed. □

As is shown in part (a) of the next example, in the general case, the preceding proposition is not valid for functionals of the form $\sum_{i=1}^{\infty}\alpha_i x_{t_i}^*$.

**Example 36.** (a) Let $\{t_n\}$ be an arbitrary sequence in $\beta\mathbb{N}$ such that $t_i \ne t_j$ for $i \ne j$ and

$$t_1 \in \overline{\{t_2, t_3, \ldots\}} \tag{47}$$

(e.g., $t_1 \in \widehat{\mathbb{N}}$ and $t_n = n-1$ if $n > 1$). Consider two arbitrary sequences $\{\alpha_n\}$ and $\{\alpha_n'\}$ in $\mathbb{R}$ satisfying

$$0 \le \alpha_n' \le \alpha_n \text{ and } \alpha_n > 0 \text{ for all } n,\ \alpha_1' > 0,\ \sum_{i=1}^{\infty}\alpha_i < \infty,\ \text{and } \lim_{n\to\infty}\frac{\alpha_n'}{\alpha_n} = 0 \tag{48}$$

(e.g., $\alpha_n' = \frac{1}{n2^n}$ and $\alpha_n = \frac{1}{2^n}$). Put $x^* = \sum_{i=1}^{\infty}\alpha_i' x_{t_i}^*$ and $y^* = \sum_{i=1}^{\infty}\alpha_i x_{t_i}^*$. Clearly, $0 \le x^* \le y^*$ and, hence, the finite additive measure $x^*$ is absolutely continuous with respect to the finite additive measure $y^*$.

If the assertion of Radon–Nikodym theorem is valid then the identity (42) holds for some sequence $w \in s$. In view of Lemma 34, we can assume $w \in \ell_\infty$. Therefore, we have the identity (43) for all $x \in \ell_\infty$, i.e., $x^*x = y^*(wx)$. Fix an index $i_0$. Since the collection $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a base for the topology on $\beta\mathbb{N}$, for an arbitrary index $n$, we find a subset $A_n$ of $\mathbb{N}$ satisfying $t_{i_0} \in \overline{A_n}$ and $(\{t_1,\ldots,t_n\}\setminus\{t_{i_0}\})\cap\overline{A_n} = \emptyset$. In view of (48), we get

$$0 \le \sum_{\substack{t_i\in\overline{A_n}\\ i\ne i_0}}\alpha_i' \le \sum_{\substack{t_i\in\overline{A_n}\\ i\ne i_0}}\alpha_i \le \sum_{i=n+1}^{\infty}\alpha_i \to 0$$

as $n \to \infty$. Then, on the one hand,

$$x^*A_n = \alpha_{i_0}' + \sum_{\substack{t_i\in\overline{A_n}\\ i\ne i_0}}\alpha_i' \to \alpha_{i_0}'$$

as $n \to \infty$, on the other hand,

$$y^*(P_{A_n}w) = \alpha_{i_0}\widehat{w}(t_{i_0}) + \sum_{\substack{t_i\in\overline{A_n}\\ i\ne i_0}}\alpha_i\widehat{w}(t_i) \to \alpha_{i_0}\widehat{w}(t_{i_0})$$

as $n \to \infty$. Whence $\alpha'_{i_0} = \alpha_{i_0}\widehat{w}(t_{i_0})$ because $x^*A_n = y^*(P_{A_n}w)$ for all $n$. Since $i_0$ is arbitrary, we get $\widehat{w}(t_i) = \frac{\alpha'_i}{\alpha_i}$ for all $i$. In particular, $\widehat{w}(t_1) \neq 0$ and $\widehat{w}(t_i) \to 0$ as $i \to \infty$.

Next, for some $n_0$ the inequality

$$|\widehat{w}(t_n)| < \frac{|\widehat{w}(t_1)|}{2} \tag{49}$$

holds for all $n \geq n_0$. Since the function $\widehat{w}$ is continuous, there exists a neighborhood $\mathcal{U}_{t_1}$ of $t_1$ such that

$$|\widehat{w}(t_1) - \widehat{w}(t)| < \frac{|\widehat{w}(t_1)|}{2} \tag{50}$$

for all $t \in \mathcal{U}_{t_1}$. Using (47), we find an index $n'$ satisfying $n' \geq n_0$ and $t_{n'} \in \mathcal{U}_{t_1}$. In view of (49) and (50), the relations

$$|\widehat{w}(t_1)| < \frac{1}{2}|\widehat{w}(t_1)| + |\widehat{w}(t_{n'})| < |\widehat{w}(t_1)|$$

hold, which is impossible.

(b) We will show that for the case of Banach–Mazur limits Radon–Nikodym theorem is not also valid. To this end, let $\{D_n\}_{n\geq 2}$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ such that $D_n \in \mathcal{D}(ac_0)$ and $\tau(\chi_{D_n}) = 1$ for all $n \geq 2$ (the existence of such sequence follows easily from, e.g., Corollary 7(a)). There exists a sequence $\{z_n^*\}_{n\geq 2}$ in ext BM satisfying $z_i^* D_j = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta. As is easy to see, $z_i^* \perp z_j^*$ for $i \neq j$. Let $z_1^*$ be a $\sigma(\ell_\infty^*, \ell_\infty)$-cluster point of the sequence $\{z_n^*\}_{n\geq 2}$. Evidently, $z_1^* \in \mathrm{BM}$ and $z_1^* D_n = 0$ for all $n \geq 2$. Again consider two sequences $\{\alpha_n\}$ and $\{\alpha'_n\}$ in $\mathbb{R}$ satisfying the relations (48) and, moreover, $\sum_{i=1}^\infty \alpha_i = 1$. Put $x^* = \sum_{i=1}^\infty \alpha'_i z_i^*$ and $y^* = \sum_{i=1}^\infty \alpha_i z_i^*$. Clearly, $0 \leq x^* \leq y^*$ and $y^* \in \mathrm{BM} \setminus \mathrm{ext\,BM}$.

Assume that the identity $x^*x = y^*(wx)$ holds for some $w \in \ell_\infty$ and for all $x \in \ell_\infty$. Using Theorem 3′(c), for an arbitrary sequence $x \in \mathcal{D}(ac_0)$, we have $\sum_{i=1}^\infty \alpha'_i z_i^* x = \sum_{i=1}^\infty \alpha_i (z_i^* w \cdot z_i^* x)$. Since the sets $D_n \in \mathcal{D}(ac_0)$, we obtain

$$\alpha'_n = \sum_{i=1}^\infty \alpha'_i z_i^* D_n = \sum_{i=1}^\infty \alpha_i (z_i^* w \cdot z_i^* D_n) = \alpha_n z_n^* w$$

for all $n \geq 2$ and, hence,

$$z_n^* w = \frac{\alpha'_n}{\alpha_n}. \tag{51}$$

Therefore, $\lim_{n\to\infty} z_n^* w = 0$ and, consequently, $z_1^* w = 0$. Using the last identity and (51), we have $\sum_{i=1}^\infty \alpha'_i = x^*\mathbf{e} = y^* w = \sum_{i=1}^\infty \alpha_i z_i^* w = \sum_{i=2}^\infty \alpha'_i$. Finally, $\alpha'_1 = 0$, a contradiction. $\square$

*It is not known if the relations* $0 \leq x^* \leq y^*$ *and* $y^* \in \mathrm{ext\,BM}$ *imply* (43).

By Bochner theorem (see [6, p. 315]), if $x^*, y^* \in \ell_\infty^*$, $y^* \geq 0$, and $x^* \ll y^*$ then for every $\epsilon > 0$ there exists a simple sequence $w$ satisfying $\|x^* - y_w^*\| < \epsilon$. The next proposition makes more precise this result.

**Proposition 37.** *Let* $x^*, y^* \in \ell_\infty^*$ *be two functionals and let* $0 \leq x^* \leq y^*$. *Then for every* $\epsilon > 0$ *there exists a simple sequence* $w$ *satisfying* $\|x^* - y_w^*\| < \epsilon$ *and* $0 \leq w \leq \mathbf{e}$.

**Proof.** First of all, we assume that $x^*$ is a component of $y^*$, i.e., $x^* \perp y^* - x^*$. There exists [3, p. 77] a collection $A_1, \dots, A_k$ of pairwise disjoint subsets of $\mathbb{N}$ which satisfies the inequality $\|x^* - \sum_{i=1}^{k} P^*_{A_i} y^*\| < \epsilon$ and so $\|x^* - P^*_B y^*\| < \epsilon$ with $B = \bigcup_{i=1}^{k} A_i$, as required.

In the general case, using Schaefer theorem [3, p. 172], we find two collections $y_1^*, \dots, y_m^*$ of components of $y^*$ and $\lambda_1, \dots, \lambda_m$ of non-negative numbers with $\sum_{i=1}^{m} \lambda_i = 1$ satisfying the inequality $\|x^* - \sum_{i=1}^{m} \lambda_i y_i^*\| < \frac{\epsilon}{2}$. As showed above, for some subsets $B_1, \dots, B_m$ of $\mathbb{N}$, we have $\|y_i^* - P^*_{B_i} y^*\| < \frac{\epsilon}{2}$ for $i = 1, \dots, m$. To finish the proof, let $w = \sum_{i=1}^{m} \lambda_i \chi_{B_i}$, and we are done. $\square$

In conclusion of this section, the author must mention that, unfortunately, he does not know such examples in the nature or in the natural science that probability models of these examples require a finite additive, i.e., discontinuous, probability measure (in particular, Banach–Mazur limit) but not countable additive.

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