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Research Article

Self-Similar Blow-Up Solutions of the KPZ Equation

Alexander Gladkov

Department of Mathematics and Mechanics, Belarusian State University, Nezavisimosti Avenue 4, 220030 Minsk, Belarus

Correspondence should be addressed to Alexander Gladkov; gladkoval@mail.ru

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Self-similar blow-up solutions for the generalized deterministic KPZ equation $u_t = u_{xx} + |u_x|^q$ with $q > 2$ are considered. The asymptotic behavior of self-similar solutions is studied.

1. Introduction

We consider the generalized deterministic KPZ equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left|\frac{\partial u}{\partial x}\right|^q \quad \text{for } (x,t) \in S_T := \mathbb{R} \times (0,T), \tag{1}$$

where $q > 2$ and $T > 0$. Equation (1) was first considered in the case $q = 2$ by Kardar et al. [1] in connection with the study of the growth of surfaces. When $q = 2$, (1) has since been referred to as the deterministic KPZ equation. For $q \neq 2$ it also called the generalized deterministic KPZ equation or Krug-Spohn equation because it was introduced in [2]. We refer to the review article [3] for references and a detailed historical account of the KPZ equation.

The existence and uniqueness of a classical solution of the Cauchy problem for (1) with $q = 1$ and initial function $u_0 \in C_0^3(\mathbb{R}^n)$ were proven in [4]. This result was extended to $u_0 \in C^2(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ and $q \geq 1$ in [5] and to $u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $q \geq 0$ in [6]. Several papers [7–11] were devoted to the investigation of the Cauchy problem for irregular initial data, namely, for $u_0 \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, or for bounded measures. The existence and uniqueness of a solution to the Cauchy problem with unbounded initial datum are proved in [12]. To confirm the optimality of obtained existence conditions, the authors of [12] analyze the asymptotic behavior of self-similar blow-up solutions of (1) for $q < 2$.

In this paper we investigate the asymptotic behavior of self-similar blow-up solutions of (1) with $q > 2$ having the form

$$u(x,t) = (T-t)^\alpha f(\xi), \quad \text{where } \xi = |x|(T-t)^\beta,\ 0 < t < T. \tag{2}$$

After substitution of (2) into (1) we find that

$$\alpha = \frac{q-2}{2(q-1)}, \quad \beta = -\frac{1}{2} \tag{3}$$

and f should satisfy the following equation:

$$f'' + \left|f'\right|^q - \frac{1}{2}\xi f' + \alpha f = 0 \quad \text{on } (0,+\infty). \tag{4}$$

We will add to (4) the following initial data:

$$f(0) = -f_0 < 0, \quad f'(0) = 0. \tag{5}$$

Put

$$C = \left[\frac{1}{q-1}\left(\frac{q-1}{q}\right)^q\right]^{1/(q-1)}. \tag{6}$$

Let us state the main result.

Theorem 1. *Let u be a self-similar blow-up solution of (1) with $q > 2$ which is defined in (2)–(5). Then*

$$\lim_{t\to T} u(x,t)(T-t)^{1/(q-1)} = C|x|^{q/(q-1)}. \tag{7}$$

A simple computation shows that Theorem 1 is a consequence of the following statement.

Theorem 2. *Let $q > 2$ and let f be a solution of problem (4), (5). Then*

$$\lim_{\xi\to\infty} \frac{f(\xi)}{\xi^{q/(q-1)}} = C. \tag{8}$$

The behavior of self-similar solutions for (1) of the type $u(x,t) = t^{\alpha} g(xt^{\beta})$ has been analyzed in [13].

2. The Proof of Theorem 2

We start with a simple result which is used later on.

Lemma 3. *Let f be a solution of problem (4), (5) defined on $[0,\bar{\xi})$. Then*

$$f'(\xi) > 0, \quad f''(\xi) > 0 \quad \text{for } \xi \in (0,\bar{\xi}). \tag{9}$$

Proof. Obviously, $f''(0) = \alpha f_0 > 0$. Therefore, by continuity, $f'' > 0$ and $f' > 0$ in some right-neighborhood of 0. Suppose that there exists ξ_0 such that $0 < \xi_0 < \bar{\xi}$, $f'' > 0$ on $[0,\xi_0)$ and $f''(\xi_0) = 0$. Then $f' > 0$ on $(0,\xi_0]$ and $f'''(\xi_0) \le 0$. From (4) we find that $f'''(\xi_0) = f'(\xi_0)/[2(q-1)] > 0$. This contradiction proves (9). □

Now we will obtain the upper bound for f'.

Lemma 4. *There exists $\xi_0 > 0$ such that*

$$f'(\xi) < \left\{\frac{\xi}{2}\right\}^{1/(q-1)} \quad \text{for } \xi \ge \xi_0. \tag{10}$$

Proof. Lemma 3 implies that $f(\xi) \to \infty$ as $\xi \to \bar{\xi}$ and that there exists unique point $\xi_0 \in (0,\bar{\xi})$ such that $f < 0$ on $(0,\xi_0)$ and $f > 0$ on $(\xi_0,\bar{\xi})$. Substituting $f'' > 0$ and $f \ge 0$ in (4) yields $f'(\xi) < \{\xi/2\}^{1/(q-1)}$ for $\xi \in [\xi_0,\bar{\xi})$. Thus, $\bar{\xi} = \infty$ and (10) holds. □

Changing variables in (4)

$$f'(\xi) = \xi^{1/(q-1)} g(t), \quad \xi = \exp t, \tag{11}$$

we get the new equation

$$g'' + \frac{3-q}{q-1}g' - \frac{q-2}{(q-1)^2}g = \left\{\frac{1}{2}g' - (g^q)' + \frac{1}{q-1}g - \frac{q}{q-1}g^q\right\}\exp(2t). \tag{12}$$

By (9), (10), and (11), there hold

$$g(t) > 0 \quad \text{for any } t \in \mathbb{R}, \tag{13}$$

$$g(t) < \left\{\frac{1}{2}\right\}^{1/(q-1)}, \quad g'(t) > -\frac{g}{q-1} \tag{14}$$

for large values of t. Put

$$C_0 = \left\{\frac{1}{q}\right\}^{1/(q-1)}, \quad C_1 = \left\{\frac{1}{2q}\right\}^{1/(q-1)}. \tag{15}$$

It is obvious that $C_0 > C_1$. Now we will establish the asymptotic behavior of $g(t)$ as $t \to +\infty$.

Lemma 5. *Assume that g is defined in (11). Then*

$$\lim_{t\to+\infty} g(t) = C_0. \tag{16}$$

Proof. From a careful inspection of (12) we conclude that a local maximum of $g(t)$ can happen only when $g(t) > C_0$.

At first we suppose that $g(t)$ does not tend to C_0 as $t \to +\infty$ and $g(t)$ is monotonic solution of (12) for large values of t. Then there exists $\bar{C} \ne C_0$ such that $\lim_{t\to\infty} g(t) = \bar{C}$. It is not difficult to show that for any $\varepsilon > 0$ there exist $A > 0$ and a sequence $\{t_k\}_{k=1}^{\infty}$ with the properties:

$$\lim_{k\to\infty} t_k = +\infty, \quad \left|g''(t_k)\right| \le A, \quad \left|g'(t_k)\right| \le \varepsilon. \tag{17}$$

Indeed, let $g' \ge 0$ for the definiteness. We suppose that $g'(t)$ is not monotonic function for large values of t since otherwise (17) is obvious. Denote by $\{\tau_k\}_{k=1}^{\infty}$ a sequence of local minima for g'. Then (17) holds for some subsequence of $\{\tau_k\}_{k=1}^{\infty}$.

Passing to the limit in (12) as $t = t_k \to +\infty$ and choosing ε in a suitable way we get that the left-hand side is bounded, while the right-hand side tends to infinity if $\bar{C} \ne 0$. Let $\bar{C} = 0$. Using (13) and (14) we conclude from (12) that

$$g'' + \frac{3-q}{q-1}g' \ge \frac{g}{3(q-1)}\exp(2t) \tag{18}$$

for large values of t. Then for large values of k (17) and (18) imply

$$g(t_k) \le \gamma \exp(-2t_k), \tag{19}$$

where positive constant γ does not depend on k. Setting $\xi_k = \exp t_k$, from (11) and (19), we get

$$f'(\xi_k) \le \gamma \xi_k^{(3-2q)/(q-1)} \tag{20}$$

that contradicts (9).

Now until the end of the proof we assume that $g(t)$ is not monotonic solution of (12) for large values of t. Suppose that $\liminf_{t\to\infty} g(t) < C_0$. Then there exist positive unbounded increasing sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ such that $t_k > s_k$,

$$g'(t) \le 0 \quad \text{for } t \in [s_k, t_k], \tag{21}$$

and $g(s_k) = C_0$, $g(t_k) = C_*$, where $C_1 < C_* < C_0$. Then

$$\begin{aligned} \frac{1}{2} g' - (g^q)' &= -q\left(g^{q-1} - C_1^{q-1}\right) g' \\ &\ge -q\left(C_*^{q-1} - C_1^{q-1}\right) g' \ge 0 \quad \text{on } [s_k, t_k]. \end{aligned} \tag{22}$$

So, (12) and (22) imply that

$$\begin{aligned} & g''(t) + \frac{3-q}{q-1} g'(t) \\ & \ge -q\left(C_*^{q-1} - C_1^{q-1}\right) g'(t) \exp(2s_k) \\ & \qquad \text{for } t \in [s_k, t_k]. \end{aligned} \tag{23}$$

Hence, integrating with respect to t from s_k to t_k, we get

$$\begin{aligned} & \left\{ g'(t) + \frac{3-q}{q-1} g(t) \right\} \Bigg|_{s_k}^{t_k} \\ & \ge q\left(C_*^{q-1} - C_1^{q-1}\right)(C_0 - C_*) \exp(2s_k). \end{aligned} \tag{24}$$

This leads to a contradiction, since (13), (14), and (21) imply that the left-hand side of the last inequality is bounded, while the right-hand side becomes unbounded as $k \to \infty$.

Let us prove that $\liminf_{t\to\infty} g(t) = C_0$. Indeed, otherwise, there exist $\varepsilon > 0$ and a sequence $\{\tau_k\}_{k=1}^{\infty}$ of local minima for g with the properties $\tau_k \to +\infty$ as $k \to +\infty$ and

$$g(\tau_k) \ge C_0 + \varepsilon. \tag{25}$$

Passing in (12) to the limit as $t = \tau_k \to +\infty$ we get a contradiction.

To end the proof we show that $\limsup_{t\to\infty} g(t) = C_0$. Otherwise, $\limsup_{t\to\infty} g(t) > C_0$. Then there exist unbounded increasing sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ such that $t_k > s_k > 2$,

$$\begin{aligned} & g'(s_k) = 0, \\ & g'(t_k) = 0, \\ & g'(t) \ge 0 \\ & g(t_k) > C_0 + \delta, \\ & |g(s_k) - C_0| < \varepsilon, \\ & \qquad \text{for } t \in [s_k, t_k], \end{aligned} \tag{26}$$

where $\delta > 0$ and

$$\varepsilon = \min\left\{ \frac{\delta}{2}, \frac{q-1}{4C_0}\delta^2, \left[1 - \left(\frac{7}{8}\right)^{1/(q-1)}\right] C_0 \right\}. \tag{27}$$

Without loss of a generality we can suppose

$$C_0 - \varepsilon < g(s_k) < C_0 \tag{28}$$

or

$$C_0 \le g(s_k) < C_0 + \varepsilon. \tag{29}$$

Let (28) be valid. If (29) is realized, the arguments are similar and simpler. Denote by $\{\bar{t}_k\}_{k=1}^{\infty}$ a sequence such that

$$\begin{aligned} & \bar{t}_k \in (s_k, t_k), \\ & g(\bar{t}_k) = C_0. \end{aligned} \tag{30}$$

Applying Hölder's inequality we derive

$$\begin{aligned} \int_{\bar{t}_k}^{t_k} g'(\tau)\, d\tau \le & \left(\int_{\bar{t}_k}^{t_k} \left(g'(\tau)\right)^2 \exp(2\tau)\, d\tau \right)^{1/2} \\ & \cdot \left(\int_{\bar{t}_k}^{t_k} \exp(-2\tau)\, d\tau \right)^{1/2} \end{aligned} \tag{31}$$

and therefore

$$\int_{\bar{t}_k}^{t_k} \left(g'(\tau)\right)^2 \exp(2\tau)\, d\tau \ge 2\delta^2 \exp(2\bar{t}_k). \tag{32}$$

We multiply (12) by $g'(t)$ and integrate after over $[s_k, t_k]$. Using (15), (26)–(28), (30), and (32) we obtain

$$\begin{aligned} & -\frac{q-2}{2(q-1)^2} g^2(t_k) \le \frac{q-3}{q-1} \int_{s_k}^{t_k} \left(g'(\tau)\right)^2 d\tau \\ & + \int_{s_k}^{t_k} \left(g'(\tau)\right)^2 \left[\frac{1}{2} - q g^{q-1}(\tau)\right] \exp(2\tau)\, d\tau \\ & + \frac{\exp(2\bar{t}_k)}{q-1} \\ & \cdot \int_{s_k}^{\bar{t}_k} \left[\frac{1}{2}\left(g^2(\tau)\right)' - \frac{q}{q+1}\left(g^{q+1}(\tau)\right)'\right] d\tau \le -\frac{1}{4} \\ & \cdot \int_{\bar{t}_k}^{t_k} \left(g'(\tau)\right)^2 \exp(2\tau)\, d\tau \\ & + \frac{\exp(2\bar{t}_k)}{q-1} \left(\frac{g^2(\tau)}{2} - \frac{q g^{q+1}(\tau)}{q+1} \right) \Bigg|_{s_k}^{\bar{t}_k} \\ & \le \left[-\frac{\delta^2}{2} + \frac{\varepsilon C_0}{q-1}\right] \exp(2\bar{t}_k) \le -\frac{\delta^2}{4} \exp(2\bar{t}_k). \end{aligned} \tag{33}$$

Passing to the limit as $k \to \infty$ we get a contradiction with (14). □

Now (8) is a simple consequence of Lemma 5 and the definition of $g(t)$.

Remark 6. Note that Theorem 2 demonstrates the optimality of Theorem 2.3 in [12]. The arguments are the same as in Remark 4.6 of that paper.

Our next result shows that (4) with initial data

$$f(0) = f_0 > 0,$$
$$f'(0) = 0 \tag{34}$$

has no global solution.

Theorem 7. *Let $q > 2$ and let f be a solution of problem (4), (34). Then there exists $\xi_\star$ such that $0 < \xi_\star < +\infty$ and $f(\xi) \to -\infty$ as $\xi \uparrow \xi_\star$.*

Proof. Suppose that problem (4), (34) has a solution f that is infinitely extendible to the right. Using the arguments of Lemma 3 we show that $f' < 0$ and $f'' < 0$ on $(0, +\infty)$. From (4) we obtain

$$f'''(\xi) < -\left(\left|f'(\xi)\right|^q\right)'. \tag{35}$$

After the integration of (35) over $[0, \xi]$ we conclude that

$$f''(\xi) < -\left|f'(\xi)\right|^q. \tag{36}$$

Integrating (36) over $[\xi_1, \xi]$ $(0 < \xi_1 < \xi)$ we infer

$$\frac{1}{(q-1)\left|f'(\xi_1)\right|^{q-1}} > \xi - \xi_1. \tag{37}$$

Passing to the limit as $\xi \to \infty$ we obtain a contradiction which proves the theorem. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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