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The Role of the Mittag-Leffler Function in Fractional Modeling

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Abstract: This is a survey paper illuminating the distinguished role of the Mittag-Leffler function and its generalizations in fractional analysis and fractional modeling. The content of the paper is connected to the recently published monograph by Rudolf Gorenflo, Anatoly Kilbas, Francesco Mainardi and Sergei Rogosin.

1. Introduction

In the paper, we would like to highlight the distinguished role of the Mittag-Leffler function and its numerous generalizations in fractional calculus and fractional modeling. Partly, the material of the paper is based on the results from the recent monograph [1].

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \text{Re}\,\alpha > 0 \tag{1}$$

The Mittag-Leffler function has been introduced to give an answer to a classical question of complex analysis, namely to describe the procedure of the analytic continuation of power series outside the disc of their convergence. Much later, theoretical applications to the study of integral equations, as well as more practical applications to the modeling of 'non-standard' processes have been found. The importance of the Mittag-Leffler function was re-discovered when its connection to fractional calculus was fully understood. Different aspects of the

distinguished role of this function in fractional theory and its applications have been described in several monographs and surveys on fractional calculus (see, e.g., [2–10]) and fractional modeling (see, e.g., [11–18]).

The paper is organized as follows. In Section 2, we briefly describe the history of the Mittag-Leffler function, paying attention mostly to the first period of the development of its theory. Section 3 is devoted to the presentation of the analytic properties of the classical Mittag-Leffler function and its direct generalizations. Theoretical applications of the Mittag-Leffler function are briefly described in Section 4. Mostly, we deal here with applications to the study of integral and differential equations of fractional order. Lastly, Section 5 is devoted to the description of a number of models in physics, mechanics, chemistry and biology in which the Mittag-Leffler function plays a crucial role. We also mention there the applications of the Mittag-Leffler function in probability theory. We have to point out that it is practically impossible to touch on all of the aspects of the theory of the Mittag-Leffler function and its generalizations, as well as their applications. The list of references is also incomplete. The interested reader can find more information in the monographs and surveys mentioned above (see also the references therein). The most extended source for such information is the recent monograph [1] (see also the papers tightly related to the topic of the monograph [19–22]).

2. History of the Mittag-Leffler Function

At the end of the 19th century, Gösta Magnus Mittag-Leffler started to work on the problem of the analytic continuation of monogenic functions of one complex variable (see, e.g., [23]). This classical question attracted the attention of the great mathematicians of that time. In particular, to solve the above problem, the construction related to the so-called Laplace–Abel integral was proposed:

$$\int\limits_0^{+\infty} e^{-\omega} F(\omega z) d\omega \tag{2}$$

where:

$$F(z) = \sum_{\nu=0}^{\infty} \frac{k_\nu}{\nu!} z^\nu, \quad \limsup_{\nu \to \infty} \sqrt[\nu]{|k_\nu|} = \frac{1}{r}$$

On the basis of his first results in the area, Mittag-Leffler made three reports in 1898 at the Royal Swedish Academy of Sciences in Stockholm. In particular, he proposed using the following generalization of the Laplace–Abel integral:

$$\int\limits_0^{+\infty} e^{-\omega} E_\alpha(\omega^\alpha z) d\omega \tag{3}$$

with E_α defined by Equation (1). The properties of the latter were studied by him in a series of five notes published in 1901–1905 (see, e.g., [1] (Ch. 2)). Nowadays, the function E_α is known as the Mittag-Leffler function.

Practically at the same time, several other functions related to the problem studied by Mittag-Leffler were introduced. Among them are the functions introduced by Le Roy [24]:

$$\sum_{n=0}^{\infty} \frac{z^n}{(n!)^p}, \quad p > 0 \tag{4}$$

by Lindelöf [25]

$$\sum_{n=0}^{\infty} \frac{z^n}{n^{\alpha n}}, \quad \alpha > 0 \tag{5}$$

$$\sum_{n=0}^{\infty} \left(\frac{z}{\log(n + 1/\alpha)} \right)^n, \quad 0 < \alpha < 1 \tag{6}$$

and by Malmquist [26]:

$$\sum_{n=2}^{\infty} \frac{z^{n-2}}{\Gamma\left[1 + \frac{n}{(\log n)^\alpha}\right]}, \quad 0 < \alpha < 1 \tag{7}$$

The direct generalization of the Mittag-Leffler function was proposed by Wiman in his work [27] on zeros of function Equation (1):

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \operatorname{Re} \alpha > 0, \ \beta \in \mathbb{C} \tag{8}$$

Later, this function was rediscovered and intensively studied by Agarval and Humbert (see, e.g., [1]). With $\beta = 1$, this function coincides with the classical Mittag-Leffler function Equation (1), *i.e.*, $E_{\alpha,1} = E_\alpha$.

The most popular and widely-applicable functions from the above-mentioned collection are the Mittag-Leffler function of one parameter Equation (1) and the Mittag-Leffler function of two parameter Equation (8).

The next period in the development of the theory of the Mittag-Leffler function is connected with increasing the number of parameters. Thus, the three-parametric Mittag-Leffler-type function was introduced by Prabhakar [28].

$$E_{\alpha,\beta}^\gamma(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{n!\Gamma(\alpha n + \beta)}, \quad \operatorname{Re} \alpha > 0, \ \operatorname{Re} \beta > 0, \ \gamma > 0 \tag{9}$$

where $(\gamma)_n = \gamma(\gamma + 1)\ldots(\gamma + n - 1)$. For $\gamma = 1$, this function coincides with two-parametric Mittag-Leffler function Equation (8), and with $\beta = \gamma = 1$, it coincides with the classical Mittag-Leffler function Equation (1), *i.e.*, $E_{\alpha,\beta,1} = E_{\alpha,\beta}$, $E_{\alpha,1,1} = E_\alpha$.

Later, in relation to the solution of a certain type of fractional differential equations, Kilbas and Saigo introduced [29] another kind of three-parametric Mittag-Leffler-type function:

$$E_{\alpha,m,l}(z) := \sum_{n=0}^{\infty} c_n z^n, \quad \operatorname{Re} \alpha > 0, m > 0, \ l \in \mathbb{C} \tag{10}$$

with:

$$c_0 = 1, \ c_n = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha[jm + l] + 1)}{\Gamma(\alpha[jm + l + 1] + 1)}$$

and parameters α, m, l are such that $\alpha[jm + l] \neq -1, -2, -3, \ldots$ for any $j = 0, 1, 2, \ldots$. With $m = 1$, this function is also reduced to two-parametric Mittag-Leffler function Equation (8), namely $E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)E_{\alpha,\alpha l+1}(z)$.

Both functions of the above type (*i.e.*, Equations (9) and (10)) are essentially used as an explicit representation of solutions to integral and differential equations of the fractional order (see [6,30]).

All of the above-mentioned Mittag-Leffler-type functions are related to the general class of special function, so-called H-functions (see [31,32]).

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z) \left[z \, \middle| \, \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^s ds \qquad (11)$$

where $\mathcal{L}$ is a suitable path on the complex plane $\mathbb{C}$, z^s is a suitably-chosen single-valued branch of the corresponding multi-valued function and:

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{A(s)B(s)}{C(s)D(s)}$$

$$A(s) = \prod_{j=1}^{m} \Gamma(b_j - \beta_j s), \ B(s) = \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)$$

$$C(s) = \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s), \ D(s) = \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)$$

The relation of the Mittag-Leffler-type functions to H-functions is due to so-called Mellin-Barnes integrals (see [33] or [1] (Appendix D)), the contour integrals:

$$I(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(s) z^{-s} ds \qquad (12)$$

with suitably-chosen contour $\mathcal{C}$ and the function f being a ratio of the products of the Γ-functions. Such integral representations constitute a basis for a unified approach for the description of the asymptotic behavior of the corresponding functions (in particular, Mittag-Leffler-type functions).

Further generalizations of the Mittag-Leffler function (multi-parametric Mittag-Leffler functions) were proposed in the same line. The detailed description of some such generalizations was discussed in [1] (Ch. 6). Among them, we have to point out the $2m$-parametric function [34] (Luchko–Kilbas–Kiryakova function; see also [35,36]):

$$E((\alpha, \beta)_m; z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \qquad (13)$$

In particular, for $m = 1$, this definition coincides with the definition of two-parametric function Equation (8), but the four-parametric function:

$$E_{\alpha_1, \beta_1; \alpha_2, \beta_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha_1 n + \beta_1)\Gamma(\alpha_2 n + \beta_2)} \qquad (14)$$

is closer by its properties to the Wright function [1] (Appendix F.2):

$$\phi(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)} \qquad (15)$$

A generalization of Prabhakar function Equation (9) was given by Shukla and Prajapati [37]:

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n} z^n}{n! \Gamma(\alpha n + \beta)}, \quad \text{Re}\,\alpha > 0, \ \text{Re}\,\beta > 0, \ \kappa > 0, \ \gamma > 0 \tag{16}$$

This definition was combined with Equation (13) by Saxena and Nishimoto [38]. As a result, the following definition of the generalized $2m + 2$-parametric function appears:

$$E_{(\alpha,\beta)_m}^{\gamma,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\gamma\kappa)_n z^n}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \tag{17}$$

Paneva-Konovska proposed another type of multi-parametric function ($3m$-parametric Mittag-Leffler function) [39].

$$E_{(\alpha_j),(\beta_j)}^{(\gamma_j),m}(z) = \sum_{n=0}^{\infty} \frac{(\gamma_1)_n \ldots (\gamma_m)_n}{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \frac{z^n}{n!} \tag{18}$$

Recently, the Le Roy-type function Equation (4), namely the function, has been introduced independently by Gerhold [40] and jointly by Garra and Polito [41]. In some sources, it is called the Gerhold–Garra–Polito function. The interest in this function is due to its certain applications. If $\gamma = m$ is a positive integer, then $F_{\alpha,\beta}^{(m)}$ is a special case of the $2m$-parametric Mittag-Leffler function Equation (13).

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\Gamma(\alpha n + \beta))^\gamma}, \quad \alpha, \beta, \gamma > 0 \tag{19}$$

There are some other generalizations of the Mittag-Leffler function in the direction of increasing of the dimension of the vectors (matrices) of parameters. Some of them are mentioned in [1] (Ch. 6).

3. Analytic Properties

First of all, we have to mention that for $\text{Re}\,\alpha > 0$ and arbitrary complex parameter β, the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is an entire function of the complex variable z.

For particular values of parameters, the Mittag-Leffler function coincides with some elementary functions. In particular,

$$E_0(z) = \frac{1}{1-z}, \ |z| < 1; \ E_{1,1}(z) = E_1(z) = \exp z; \ E_{1,2}(z) = \frac{e^z - 1}{z} \tag{20}$$

$$E_{2,1}(z^2) = E_2(z^2) = \cosh z; \ E_{2,1}(-z^2) = E_2(-z^2) = \cos z \tag{21}$$

$$E_{2,2}(z^2) = \frac{\sinh z}{z}; \ E_{2,2}(-z^2) = \frac{\sin z}{z}; \ E_4(z) = \frac{1}{2}\left[\cos z^{1/4} + \cosh z^{1/4}\right] \tag{22}$$

$$E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right] \tag{23}$$

$$E_{1/2}(\pm z^{1/2}) = e^z\left[1 + \text{erf}(\pm z^{1/2})\right] = e^z\text{erfc}(\pm z^{1/2}) \tag{24}$$

Below in Table 1, we present a few properties of the best known Mittag-Leffler-type functions, namely $E_\alpha(z)$, $E_{\alpha,\beta}(z)$, $E_{\alpha,\beta}^\gamma$, $E_{\alpha,m,l}(z)$. The properties that we discuss here are the following:

the order ρ of an entire function $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$:

$$\rho = \limsup_{k \to \infty} \frac{k \log k}{\log \frac{1}{|c_k|}} \qquad (25)$$

- the type σ of an entire function $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ of the order ρ:

$$(\sigma e \rho)^{1/\rho} = \limsup_{k \to \infty} \left(k^{1/\rho} \sqrt[k]{|c_k|} \right) \qquad (26)$$

- the Laplace transform of a function $f(t)$:

$$f(t) \div (\mathcal{L}f)(s) = \int_{0}^{+\infty} f(t) e^{-st} dt, \quad \mathrm{Re}\, s > s_0 \qquad (27)$$

- the Riemann-Liouville fractional integral $\mathcal{I}_{0+}^{\alpha}$ of a function $f(t)$:

$$\left(\mathcal{I}_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t) dt}{(x-t)^{1-\alpha}}, \quad x > 0 \qquad (28)$$

- the Riemann-Liouville fractional derivative $\mathcal{D}_{0+}^{\alpha}$ of a function $f(t)$:

$$\left(\mathcal{D}_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{0}^{x} \frac{f(t) dt}{(x-t)^{1-m+\alpha}}, \quad x > 0, \; m-1 < \alpha < m \qquad (29)$$

Table 1. Properties of the functions $E_{\alpha}(z)$, $E_{\alpha,\beta}(z)$, $E_{\alpha,\beta}^{\gamma}(z)$, $E_{\alpha,m,l}(z)$.

f	**ρ**	**σ**	**$\mathcal{L}\,\mathbf{f}$**	**$\left(\mathcal{I}_{0+}^{\alpha}\boldsymbol{f}\right)$**	**$\left(\mathcal{D}_{0+}^{\alpha}\boldsymbol{f}\right)$**
E_{α}	$\frac{1}{\alpha}$	1	$f = E_{\alpha}(\lambda t^{\alpha})$ $f \div \frac{s^{\alpha-1}}{s^{\alpha}\mp 1}$	$f = E_{\alpha}(\lambda t^{\alpha})$ $\frac{1}{\lambda}\{f(x) - 1\}$	$f = E_{\alpha}(\lambda t^{\alpha})$ $(0 < \alpha < 1)$ $\frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \lambda f(x)$
$E_{\alpha,\beta}$	$\frac{1}{\alpha}$	1	$f = t^{\beta-1} E_{\alpha,\beta}(\lambda t^{\alpha})$ $f \div \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$	$f = t^{\alpha-1} E_{\alpha,\beta}(\lambda t^{\alpha})$ $\frac{f(x)}{\lambda} - \frac{x^{\alpha-1}}{\lambda\Gamma(\beta)}$	$f = t^{\beta-1} E_{\alpha,\beta}(\lambda t^{\alpha})$ $(\alpha > 0, \beta > \alpha+1)$ $\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + \lambda f(x)$
$E_{\alpha,\beta}^{\gamma}$	$\frac{1}{\alpha}$	1	$f = t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\lambda t^{\alpha})$ $f \div \frac{s^{-\beta}}{(1-\lambda s^{-\alpha})^{\gamma}}$	$f = t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\lambda t^{\alpha})$ $x^{\beta+\alpha-1} E_{\alpha,\beta+\alpha}^{\gamma}(\lambda t^{\alpha})$	$f = t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\lambda t^{\alpha})$ $(\beta > \alpha > 0)$ $x^{\beta-\alpha-1} E_{\alpha,\beta-\alpha}^{\gamma}(\lambda t^{\alpha})$
$E_{\alpha,m,l}$	$\frac{1}{\alpha}$	$\frac{1}{m}$	The result is not known	$f = t^{\alpha l} E_{\alpha,m,l}(\lambda t^{\alpha m})$ $x^{\alpha(-m+1)} f(x) - x^{\alpha(l-m+1)}$	$f = t^{\alpha-j} E_{\alpha,m,l}(\lambda t^{\alpha m})$ $\alpha > 0, m > 0,$ $j = -\alpha(l-m)$ $\lambda x^{\alpha l} E_{\alpha,m,l}(\lambda t^{\alpha m})$

Several other analytic properties are discussed in [1] (Ch. 3–6), in particular different types of recurrence relations, integral and differential properties. The study of zeros' distribution is presented there, too (see also [42]). These results are helpful for the investigation of certain inverse scattering problems, as well as other problems from the operator theory. For calculation of the Mittag-Leffler function, one can use, e.g., [43]. Asymptotic properties of the Mittag-Leffler-type functions show their place in the whole theory of special functions (see also [30]). Complete monotonicity of the Mittag-Leffler function (*i.e.*, alternating of the signs of the successive derivatives) is applied to the study of the Lévy stable distributions; relations to different kinds integral transforms are used as the solution to integral (see also [4]) and differential equations (see also [2,6]).

4. Applications to Fractional Order Equations

The most simple integral equation of the fractional order, namely the Abel integral equation of the first kind, was investigated by Abel himself. The Abel integral equation of the second kind was studied by Hille and Tamarkin [44]. In this paper, the solution was for the first time represented via the Mittag-Leffler function. The interested reader is referred to [4], and to [1] for historical notes and the detailed analysis with applications.

It is well known that Niels Henrik Abel was led to his famous equation by the mechanical problem of the tautochrone, that is by the problem of determining the shape of a curve in the vertical plane, such that the time required for a particle to slide down the curve to its lowest point is equal to a given function of its initial height (which is considered as a variable in an interval $[0, H]$). After appropriate changes of variables, he obtained his famous integral equation of the first kind with $\alpha = 1/2$. He did, however, solve the general case $0 < \alpha < 1$. As a special case, Abel discussed the problem of the isochrone, in which it is required that the time of sliding down is independent of the initial height. Already in his earlier publication, he recognized the solution as a derivative of non-integer order.

The Abel integral equations occur in many situations where physical measurements are to be evaluated. In many of these cases, the independent variable is the radius of a circle or a sphere, and only after a change of variables, the integral operator has the form $\mathcal{I}^\alpha$, usually with $\alpha = 1/2$, and the equation is of the first kind. For instance, there are applications in the evaluation of spectroscopic measurements of cylindrical gas discharges, the study of the solar or a planetary atmosphere, the investigation of star densities in a globular cluster, the inversion of travel times of seismic waves for the determination of terrestrial sub-surface structure and in the solution of problems in spherical stereology. Descriptions and analysis of several problems of this kind can be found in the books by Gorenflo and Vessella [4].

Another field in which the Abel integral equations or integral equations with more general weakly singular kernels are important in the study of inverse boundary value problems in partial differential equations, in particular the parabolic ones, in which the independent variable has the natural meaning of time.

The paper of O'Shaughnessay (1919) was probably the first where the methods for solving the differential equation of the half-order:

$$(\mathcal{D}^{1/2}y)(x) = \frac{y}{x} \tag{30}$$

were considered. Two solutions of such an equation:

$$y(x) = x^{-1/2}e^{-1/x} \tag{31}$$

and a series solution:

$$y(x) = 1 - i\sqrt{\pi}x^{-1/2}e^{-1/x} + x^{-1/2}e^{-1/x}\int_{-\infty}^{x} t^{3/2}e^{1/t}dt = \tag{32}$$

$$= 1 - i\sqrt{\pi}x^{-1/2} - 2x^{-1} + i\sqrt{\pi}x^{-3/2} + \cdots$$

were suggested by O'Shaughnessay and discussed later by Post (see [1] (Ch. 7)). Their arguments were formal and based on an analogy of the Leibniz rule, which for the Riemann–Liouville fractional derivative has the form:

$$\left(\mathcal{D}_{0+}^{\alpha}(fg)\right)(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)k!}(\mathcal{D}_{0+}^{\alpha-k}f)(x)g^{(k)}(x)$$

As was proven later (see, e.g., [45], pp. 195–199), Equation (31) is really the solution to Equation (30) with the Riemann–Liouville fractional derivative $\mathcal{D}_{0+}^{1/2}y$. As for Equation (32), it is not a solution of Equation (30) with $\mathcal{D}^{\alpha} = \mathcal{D}_{a+}^{\alpha}$, a being any real constant, because O'Shaughnessay and Post made a mistake while using the relation for the composition of $\mathcal{I}^{1/2}\mathcal{D}^{1/2}y$. Such a relation for the Riemann–Liouville derivative $\mathcal{D}_{a+}^{\alpha}y$ has the form:

$$(\mathcal{I}_{a+}^{\alpha}\mathcal{D}_{a+}^{\alpha}y)(x) = y(x) - \sum_{k=1}^{n} B_k \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}$$

where:

$$B_k = y_{n-\alpha}^{(n-k)}(a), \ y_{n-\alpha}(x) = (\mathcal{I}_{a+}^{n-\alpha}y)(x) \ (\alpha \in \mathbb{C}, \ n = [\mathrm{Re}(\alpha)]+1)$$

in particular,

$$(\mathcal{I}_{a+}^{\alpha}\mathcal{D}_{a+}^{\alpha}y)(x) = y(x) - B\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \ B = y_{1-\alpha}(a)$$

for $0 < \mathrm{Re}(\alpha) < 1$. O'Shaughnessay and Post have applied the later formula for $\alpha = 1/2$ and $a = 0$ by considering the constant B instead of the monomial $Bx^{-1/2}$.

Mandelbroit (1925) arrived at a differential equation of fractional order when he investigated an extremum problem for the functional $\int_0^1 F[\mathcal{D}_{a+}^{\alpha}y(x); x]dx$ with the Riemann–Liouville fractional derivative $\mathcal{D}_{a+}^{\alpha}y(x)$. He has assumed that the corresponding variations are equal to zero and obtained the differential equation with Cauchy conditions:

$$dF \equiv F[\mathcal{D}_{a+}^{\alpha}y(x); x] = 0, \ y^{(k)}(a) = b_k \ (k = 1, 2, \cdots, n)$$

One of the first investigations of differential equations of fractional order was made by Barrett (1954). He considered differential equations with the fractional derivative of the Riemann–Liouville-type of arbitrary order α, $\mathrm{Re}\,\alpha > 0$, where n boundary conditions ($n = \mathrm{Re}\,\alpha + 1$) in the form of the values at the initial point of the fractional derivatives of order $\alpha - k$, $k = 1, 2, \ldots, n$ are posed. It was shown that in a suitable class of functions, the solution is unique and is represented via the Mittag-Leffler function.

One of the leading methods for linear fractional ordinary differential equations is their (equivalent) reduction to certain Volterra integral equations in proper functional spaces. An extended technique based on this method in the application to different kinds of fractional differential equations is presented in the monograph [6] (see also [1] (Ch. 7)). Other methods proposed for explicitly solving ordinary fractional differential equations and discussed in [6] are the compositional method, the operational method (see also [46]) and the integral transforms method (see also [47]).

It was noted, in particular, that different types of fractional derivatives involved in the equations lead to different kinds of initial conditions, e.g., if a differential equation contains the Riemann–Liouville fractional derivative, then the natural initial conditions are so-called Cauchy-type conditions, *i.e.*, conditions of the type, but in the case of the Caputo derivatives, it is natural to pose the standard Cauchy conditions.

$$\mathcal{D}_{a+}^{\beta_k}(a+) = b_k$$

It was probably Dzherbashian (1970) who first considered the Dirichlet-type problems for the integro-differential equations of fractional order.

Classification of linear and non-linear partial differential equations of fractional order is still far from being completed. Several results for partial differential equations are described in [48] (see also [6]). Among these results, we mention the pioneering work by Gerasimov (1948) and recent books [2,15]. Anyway, this area is rapidly growing, since most of the results are related to different types of applications. It is impossible to describe all existing results. Partly, they are presented in [1]. We also note that many authors have applied methods of fractional integro-differentiation to construct solutions to ordinary and partial differential equations, to investigate integro-differential equations and to obtain a unified theory of special functions.

5. Mittag-Leffler Functions in Fractional Modeling

The branch of fractional modeling can be formally divided into two sub-directions, namely the creation and study of deterministic models (see [1] (Ch. 8)(and stochastic models ([1] (Ch. 9)).

The first sub-direction is more developed. Anyway, it should be noted that there is a large amount of articles where the models are introduced formally. Among the models dealing with more practical applications, we can point out those related to fractional viscoelasticity (see [15]). Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory (see [49]). During the twentieth century, a number of authors have (implicitly or explicitly) used the fractional calculus as an empirical method for describing the properties of viscoelastic materials. Such investigations are based on the classical works by Scott-Blair, Gemant, Gerasimov and Rabotnov (see [15,17,50]). The beginning of the modern applications of fractional calculus in linear viscoelasticity is generally attributed to the 1979 Ph.D. thesis by Bagley (under supervision of Prof. Torvik), followed by a number of relevant papers. However, for the sake of completeness, one would recall also the 1970 PhD thesis of Rossikhin under the supervision of Prof. Meshkov and the 1971 PhD thesis of Mainardi under the supervision of Prof. Caputo.

As applications in physics and chemistry, we would like to quote the contributions by Kenneth S. Cole (1933), quoted in connection with nerve conduction, and by de Oliveira Castro (1939), Kenneth S.

Cole and Robert H. Cole (1941–1942) and Gross (1947) in connection with dielectric and mechanical relaxation, respectively (see also [18,51]). Subsequently, in 1971, Caputo and Mainardi (see [15]) proved that the Mittag-Leffler function is present whenever derivatives of fractional order are introduced into the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of the Mittag-Leffler function to fractional viscoelastic models (see [15,52–54] and the references therein). We also mention here a growing number of applications of the Mittag-Leffler function in control theory (see [11,13,16].

Stochastic modeling, which uses the fractional calculus approach, as well as the machinery of the Mittag-Leffler functions, is connected mainly with the concept of the continuous time random walk (CTRW) (see [1,12]).

A fractional generalization of the Poisson probability distribution was presented by Pillai in 1990 in his pioneering work. He introduced the probability distribution (which he called the Mittag-Leffler distribution) using the complete monotonicity of the Mittag-Leffler function. The concept of a geometrically infinitely-divisible distribution was introduced in 1984 by Klebanov, Maniya and Melamed. Later, in 1995, Pillai introduced a discrete analogue of such a distribution (*i.e.*, the discrete Mittag-Leffler distribution). Another possible variant of the generalizations of the Poisson distribution is that introduced by Lamperti in 1958, whose density has the expression:

$$f_{X_\alpha}(y) = \frac{\sin \pi \alpha}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos \pi \alpha + 1}$$

The concept of renewal process has been developed as a stochastic model for describing the class of counting processes for which the times between successive events are independent identically distributed (i.i.d.) non-negative random variables, obeying a given probability law. These times are referred to as waiting times or inter-arrival times. The process of the accumulation of waiting times is inverse to the counting number process, called the Erlang process.

The Mittag-Leffler function appears also in the solution of the fractional master equation. Such an equation characterizes the renewal processes with reward modeling by the random walk model known as continuous time random walk. In that, the waiting time is assumed to be a continuous random variable. The name CTRW became popular in physics after publication in the 1960s by Montroll, Weiss and Scher, the celebrated series of papers on random walks to model diffusion processes on lattices. The basic role of the Mittag-Leffler waiting time probability density in time fractional continuous time random walk became well known since the fundamental paper by Hilfer and Anton (1995). Earlier, this conception was used in the theory of thinning (rarefaction) of a renewal process under power law assumptions. CTRWs are rather good and general phenomenological models for diffusion, including anomalous diffusion, provided that the resting time of the walker is much greater than the time it takes to make a jump. In fact, in the formalism, jumps are instantaneous. In more recent times, CTRWs were applied back to economics and finance. It should be noted, however, that the idea of combining a stochastic process for waiting times between two consecutive events and another stochastic process, which associates a reward or a claim to each event, dates back at least to the first half of the twentieth century with the so-called Cramér–Lundberg model for insurance risk. In a probabilistic framework, we now find it more appropriate to refer to all of these processes as compound renewal processes. An alternative renewal process called the Wright process (close to the Mittag-Leffler process) was investigated by Mainardi

et al. (see [55]) as a process arising by discretization of the stable subordinator. This approach is based on the concept of the extremal Lévy stable density (Lévy stable processes are widely discussed in several books on probability theory). For the study of the Wright processes, an essential role is played by the so-called M-Wright function, a variant of the Mittag-Leffler function (see, e.g., [15]). A scaled version of this process has been used by Barkai (2002) for approximating the time-fractional diffusion process directly by a random walk subordinate to it (executing this scaled version in natural time), and he has found rather poor convergence in refinement.

Several other stochastic models in which the Mittag-Leffler function is used as a basic concept and technique are discussed in [1] (Ch. 9).

6. Conclusions

In this paper, we briefly touch on a number of results related to the celebrated Mittag-Leffler function. Partly, these results are presented in the recent monograph [1]. This topic still needs a more extended and careful description. It is important, since the role of the Mittag-Leffler function as the queen function of fractional calculus has not been overestimated up to now.

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Conflicts of Interests

The author declares no conflict of interest.

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