INVESTIGATION OF CONDITIONS FOR ASYMPTOTIC NORMALITY OF SPECTRAL ESTIMATES

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Abstract

We study the question of validity of central limit theorems for empirical spectral functionals of stationary stochastic processes and fields.

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Given observations of a real-valued stationary random field $X(t)$, $t \in Z^d$, on a sequence of hypercubes $L_T = [-T, T]^d = \{t \in Z^d : -T \leq t_i \leq T,\ i = 1, ..., d\}$, we consider spectral functionals

$$J_T(\varphi) = \int_S I_T^h(\lambda)\,\varphi(\lambda)\,d\lambda,$$

where $S = (-\pi, \pi]^d$ and $I_T^h(\lambda)$ is the second-order periodogram based on the tapered values $\{h_T(t)\,X(t),\ t \in L_T\}$. We suppose that the taper $h_T(t)$ factorises and satisfies some standard conditions.

Suppose that all order moments exist and the field $X(t)$ has spectral densities of all orders $f_k(\lambda_1, ..., \lambda_{k-1}) \in L_1(S^{k-1})$, $k = 2, 3, ...$

To derive central limit theorems for the spectral functionals $J_T(\varphi)$ we can use the approach based on calculation and evaluation of their cumulants. Within this approach conditions for the asymptotic normality of (appropriately normalized) functional $J_T(\varphi)$ can be stated in terms of conditions on spectral densities $f_k(\lambda_1, ..., \lambda_{k-1})$ and functions φ, in particular, under the conditions of their integrability.

For Gaussian and linear fields, it is possible to state central limit theorems for the functionals $J_T(\varphi)$ under the conditions of integrability of the following form: the spectral density of the field $f(\lambda) \in L_p$ and $\varphi(\lambda) \in L_q$, where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$.

We state next, for long-range dependent Gaussian random fields, the asymptotic normality result for the functional $J_T(\varphi)$ under the conditions prescribing behavior of the spectral density at the point of singularity.

Theorem 1. *Let $X(t)$, $t \in Z^d$, be a zero-mean stationary Gaussian random field with spectral density $f(\lambda)$, $\lambda \in S$, such that for some $0 < \alpha_i < 1$, $i = 1, .., d$, $f(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{-\alpha_i})$ as $\lambda_i \to 0$, and $\varphi(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{\alpha_i})$ as $\lambda_i \to 0$. The sets of discontinuities of functions $f(\lambda)$ and $\varphi(\lambda)$ have Lebesgue measure zero, and these functions are bounded for $\delta \leq |\lambda| \leq \pi$ for all $\delta > 0$. Then*

$$T^{d/2}(J_T(\varphi) - EJ_T(\varphi)) \xrightarrow{D} N(0, \sigma^2) \quad \text{as } T \to \infty, \tag{1}$$

where $\sigma^2 = 2(2\pi)^d e(h) \int_S f^2(\lambda)\varphi^2(\lambda)d\lambda$, and the factor $e(h)$ depends on the taper function.

We give conditions on the taper under which the statement (1) can be strengthen to the form $T^{d/2}(J_T(\varphi) - J(\varphi)) \xrightarrow{D} N(0,\sigma^2)$ as $T \to \infty$, where $J(\varphi) = \int_S f(\lambda)\,\varphi(\lambda)\,d\lambda$.

We state also the conditions of asymptotic normality for nonlinear functionals of the periodogram, namely, for the spectral functionals of powers of a periodogram: $J_T^{(k)}(\varphi) = \int_S \varphi(\lambda)\left(I_T^h(\lambda)\right)^k d\lambda$.

The main analytic tool used in the proofs in order to evaluate of the asymptotic behaviour of cumulants of spectral functionals is the generalized Hölder inequality.

Statistical analysis based on second-order information (covariance and spectrum) is not always sufficient or not sufficiently good in some situations and one needs to consider higher-order information, higher-order moments/cumulants and higher-order spectra. Statistical techniques based on higher-order moments and spectra are of great demand in many fields of applications, which include: geophysics, astronomy, oceanography, communications, image processing, fluid mechanics, plasma physics, astrophysics, turbulence, economics and finance. In particular, some motivations behind the use of higer-order spectra in signal processing are: to detect and characterize nonlinearities, to detect signal from Gaussian and non-Gaussian noise. The investigation of various statistical problems in the mentioned areas leads to the consideration of functionals of higher order spectral densities

$$J_k(\varphi) = \int_S \varphi_k(\lambda_1, ..., \lambda_{k-1})\, f_k(\lambda_1, ..., \lambda_{k-1})\, d\lambda_1 \cdots d\lambda_{k-1},$$

for an appropriate function φ_k.

We study the estimators for $J_k(\varphi)$ based on tapered periodograms of higher orders. We introduce also some modifications of such estimators intended to reduce a bias of estimators. And within one more approach we construct the estimators for higher order spectral functionals recursively.

We consider the application of asymptotic results for empirical spectral functionals to the problems of parameter estimation in the spectral domain. Central limit theorems for spectral functionals serve as the main tool to derive asymptotic properties of so-called minimum contrast estimators.

The presentation is partly based on the results obtained jointly with N. Leonenko and F. Avram. Applications for minimum contrast parameter estimation for several models of long-range dependent Gaussian fields are presented in the paper [1].

References

[1] Alomari H.M., Frías M.P., Leonenko N.N., Ruiz-Medina M.D., Sakhno L., and Torres A. (2017). Asymptotic properties of parameter estimates for random fields with tapered data. *Electronic Journal of Statistics*. Vol. **11**, pp. 3332-3367.