THE PROBLEM OF THE REALITY OF THE LAPLACIAN SPECTRUM OF DIGRAPHS

R. Agaev

Institute of Control Sciences of the Russian Academy of Sciences, Moscow, Russia

e-mail: arpoye@rambler.ru

A Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{n\times n}$ has nonpositive off-diagonal entries and zero row sums. Every nonsymmetric Laplacian matrix is associated with a directed graph $\Gamma(V, E)$ with vertex set $V = \{1, \dots, n\}$ and arc set E. In this paper we investigate the Laplacian spectrum of the digraphs that consist of two contradirectional Hamiltonian cycles from one of which one or two arcs were removed. The characteristic polynomials for these matrices are studied by means of the polynomials $Z_n(x)$ that satisfy the recurrence relation $Z_n(x) = (x-2)Z_{n-1}(x) - Z_{n-2}(x)$ with the initial conditions $Z_0(x) \equiv 1$ and $Z_1(x) \equiv x - 1$. We show that $Z_n(x)$ and Chebyshev polynomials of the second kind $P_n(x)$ are related by $Z_n(x) = P_{2n}(\sqrt{x})$.

Keywords: Laplacian matrix; Laplacian spectrum; Chebyshev polynomials; Directed graphs.

1. INTRODUCTION

In this paper we consider nonsymmetric Laplacian matrices $L = (l_{ij})$ such that $l_{ij} \in \{0, -1\}$ for $i \neq j$ and $l_{ii} = -\sum_{j\neq i} l_{ij}$ for all $i = 1, ..., n$. Every matrix of this kind can be associated with an unweighted directed graph $\Gamma(V, E)$ with no loops. In this case $l_{ij} = -1$ iff $(i, j) \in E(\Gamma)$.

The Chebyshev polynomial of the second kind $P_n(x)$ on $[-2, 2]$ is a polynomial of degree n defined by

$$P_n(x) = \frac{\sin((n+1)\arccos\frac{x}{2})}{\sqrt{1 - \frac{x^2}{4}}}, \tag{1}$$

where $\frac{x}{2} = \cos\varphi$ and $\varphi \in]0, \pi[$.

By (1)

$$P_n(x) = \frac{\sin(n+1)\varphi}{\sin\varphi}. \tag{2}$$

It is known that $P_n(x)$ can be expressed as follows:

$$P_n(x) = U_n(x/2) = \sum_{i=0}^{[n/2]} (-1)^i C_{n-i}^i x^{n-2i} \tag{3}$$

and satisfies the recurrence relation

$$P_n(x) = xP_{n-1}(x) - P_{n-2}(x) \tag{4}$$

with the initial conditions $P_0(x) \equiv 1$ and $P_1(x) \equiv x$. By (2) the zeros of $P_n(x)$ are readily determined from the zeros of $\sin(n+1)\varphi$:

$$x_k = 2\cos\frac{\pi k}{n+1}, \quad k = 1, \dots, n. \tag{5}$$

Let $A_n = (a_{ij})$ be the tridiagonal matrix

$$A_n = \begin{pmatrix} \lambda-2 & 1 & & & & & \\ 1 & \lambda-2 & 1 & & & & \\ & & \dots & & & & \\ & & & \dots & & & \\ & & & & \dots & & \\ & & & & 1 & \lambda-2 & 1 \\ & & & & & 1 & \lambda-1 \end{pmatrix},$$

where λ is a parameter. In particular, $\det A_1 = (\lambda-1)$. Let $\det A_0 = 1$. Using the cofactor expansion along the first row for $\det A_n$ we get

$$\det A_n = (\lambda-2)\det A_{n-1} - \det A_{n-2}, \quad n \geq 2.$$

It is obvious that for every $n = 1, 2, \dots$ the determinant of A_n is the polynomial $Z_n(\lambda)$ defined by the recurrence relation

$$Z_n(\lambda) = (\lambda-2)Z_{n-1}(\lambda) - Z_{n-2}(\lambda) \tag{6}$$

with the initial conditions $Z_0(\lambda) \equiv 1$ and $Z_1(\lambda) \equiv \lambda - 1$.

The following lemmas will by used to prove the main theorems.

Lemma 1. $P_{2n}(x) = Z_n(x^2)$.

Corollary to Lemma 1. 1. *The polynomial $Z_n(x)$ has the form*

$$Z_n(x) = \sum_{i=0}^{n}(-1)^i C_{2n-i}^i x^{n-i}. \tag{7}$$

2. *The set*

$$\left\{4\cos^2\frac{\pi k}{2n+1} \,|\, k = 1, \dots, n\right\}$$

is that of the zeros of the polynomial $Z_n(x)$ and it coincides with the set of squares of the zeros of the polynomial $P_{2n}(x)$.

To make sure that the item 1 is true it is sufficient to compare (7) with (3).
To prove item 2, observe that the zeros of $P_{2n}(x)$ are $2\cos\frac{\pi k}{2n+1}$, $k = 1, \dots, 2n$ and they can be presented as $\pm 2\cos\frac{\pi k}{2n+1}$, $k = 1, \dots, n$. The set of the squares of these numbers

coincides with the set $\{4\cos^2\frac{\pi k}{2n+1} | k = 1, \ldots, n\}$. It is obvious that this set contains all zeros of $Z_n(x)$ in the $[0, 4]$.

By $P_{2n}(\sqrt{x}) = Z_n(x)$ and (1), the polynomial $Z_n(x)$ has the following trigonometric representation:

$$Z_n(x) = \frac{\sin(2n+1)\varphi}{\sin\varphi},$$

where $x = 4\cos^2\varphi, \varphi \in]0, \frac{\pi}{2}], x \in [0, 4[$. So $Z_n(x)$ can be called the **Chebyshev polynomial of the second kind scaled on** $[0, 4]$.

Lemma 2. *On the half-open interval* $]0, \frac{\pi}{2}]$ *the roots of*

$$Z_n(x) + (-1)^p = 0$$

are $4\cos^2\frac{\pi k}{2n+1+(-1)^{k+p}}$, $k = 1, \ldots, n$, *where* $p \in \{0, 1\}$.

2. MAIN THEOREMS

Let L_1 be the Laplacian matrix of the digraph that consists of two contradirectional Hamiltonian cycles from one of which arc $(n, 1)$ is removed. In other words, this digraph consists of two contradirectional routs and an arc connecting their ends.

$$L_1 = \begin{pmatrix} 2 & -1 & & & & & -1 \\ -1 & 2 & -1 & & & & \\ & & \cdots & & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix},$$

Theorem 1. *The characteristic polynomial of* L_1 *is* $\Delta_{L_1}(\lambda) = Z_n(\lambda) - (-1)^n$. *The zeros of* $\Delta_{L_1}(\lambda)$ *are* $4\cos^2\frac{\pi k}{2n+1-(-1)^{k+n}}$, $k = 1, \ldots, n$.

Proof. By expanding $\lambda I - L_1$ with respect to the first row we have:

$$|\lambda I - L_1| = (\lambda - 2)Z_{n-1}(\lambda) - Z_{n-2}(\lambda) - (-1)^n.$$

It follows from (6) that

$$|\lambda I - L_1| = \Delta_{L_1}(\lambda) = Z_n(\lambda) - (-1)^n.$$

It follows from Lemma 2 that the roots of the characteristic polynomial $\Delta_{L_1}(\lambda)$ of L_1 are $4\cos^2\frac{\pi k}{2n+1-(-1)^{k+n}}$, $k = 1, \ldots, n$. □

Now consider the Laplacian matrix L_2 of the digraph that differs from the previous one by removing the arc $(i, i+1)$:

$$1 \quad 2 \quad \ldots \quad i \quad i+1 \quad \ldots \quad n-1 \quad n$$

$$L_2 = \begin{array}{c} \begin{pmatrix} 2 & -1 & & & & & & -1 \\ -1 & 2 & & & & & & \\ & & \cdots & & & & & \\ & & & 1 & 0 & & & \\ & & & -1 & 2 & & & \\ & & & & & \cdots & & \\ & & & & & & 2 & -1 \\ & & & & & & -1 & 1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ \cdots \\ i \\ i+1 \\ \cdots \\ n-1 \\ n \end{array}$$

The problem of the reality of the eigenvalues of this matrix is related to the properties of the product of Chebyshev polynomials of the second kind. We give some properties of this product in the following two lemmas.

Theorem 2. 1. *The characteristic polynomial of* L_2 *is*

$$\Delta_{L_2}(\lambda) = Z_i(\lambda)Z_{n-i}(\lambda) - (-1)^n.$$

2. *If n is even, then all eigenvalues of L_2 are real if and only if $i = \frac{n}{2}$, in which case they are* $4\cos^2\frac{\pi k}{n}$, $4\cos^2\frac{\pi k}{n+2}$, $k = 1, \dots, \frac{n}{2}$.

3. *If n is odd, then all eigenvalues of L_2 are real if and only if $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$; in either case they are* 0 (*with multiplicity* 1) *and* $4\cos^2\frac{\pi k}{n+1}$, $k = 1, \dots, \frac{n-1}{2}$ (*with multiplicity* 2 each).

Proof. 1. Expanding $\lambda I - L_2$ with respect to the first row and substituting the expression (6) for $(\lambda - 2)Z_{i-1}(\lambda) - Z_{i-2}(\lambda)$ we have:

$$\begin{aligned} \Delta_{L_2}(\lambda) &= (\lambda - 2)Z_{i-1}(\lambda)Z_{n-i}(\lambda) - Z_{i-2}(\lambda)Z_{n-i}(\lambda) - (-1)^n \\ &= Z_i(\lambda)Z_{n-i}(\lambda) - (-1)^n. \end{aligned}$$

Let

$$x_1^{(m)} = 4\cos^2\frac{\pi m}{2m+1} \quad \text{and} \quad x_2^{(m)} = 4\cos^2\frac{\pi(m-1)}{2m+1}$$

be the smallest and the second smallest roots of $Z_m(x)$, respectively,

$$u_1^{(m)} = 4\cos^2\frac{\pi m}{2(m+1)} \quad \text{and} \quad u_2^{(m)} = 4\cos^2\frac{\pi(m-1)}{2m}$$

the roots of $Z_m + (-1)^m = 0$ for $k = m$ and $k = m - 1$, respectively.

Consider the polynomials $Z_i(x)$ and $Z_j(x)$ with arbitrary natural i and j.

We need the following two lemmas.

Lemma 3. 1. *If* $u_1^{(i)} < x_2^{(j)}$, *then*

$$i > \frac{2j-2}{3}.$$

2. *If* $0 < i < j$, *then* $u_1^{(i)} \geq u_2^{(j)}$.

3. *The inequality*

$$|Z_i(x)Z_j(x)| < 1 \tag{8}$$

holds on $]0, \max(x_1^{(i)}, u_2^{(j)})]$.

4. *If* $u_1^{(i)} < x_2^{(j)}$ *and* $0 < i < j-1$, *then* (8) *holds for all* $x \in [u_1^{(i)}, x_2^{(j)}]$.

Lemma 4. 1. *If* $i+j$ *is even, then* $Z_i(x)Z_j(x) - 1 = 0$ *has only real roots if and only if* $i = j$. *In this case, the roots are* $4\cos^2\frac{\pi k}{i}$, $4\cos^2\frac{\pi k}{i+2}$, $k = 1, \ldots, i$.

2. *If* $i < j$ *and* $i+j$ *is odd, then* $Z_i(x)Z_j(x) + 1 = 0$ *has only real roots if and only if* $i = j-1$. *In this case,*

$$Z_i(x)Z_j(x) + 1 = P_{i+j}^2\left(\sqrt{x}\right)$$

holds and the roots are 0 *with multiplicity* 1 *and* $4\cos^2\frac{\pi k}{2i}$, $k = 1, \ldots, i$ *with multiplicity* 2.

Items 2 and 3 of Theorem 2 follow from items 1 and 2 of Lemma 4, respectively. ☐

3. CONCLUSIONS

Let L_n^c be the Laplacian matrix of the chain with n vertices. By suitable indexing of the vertices we can present $\lambda I - L_n^c$ as the matrix different from A_n (defined above) only in the $(1,1)$ entry, which for $\lambda I - L_n^c$ is $\lambda - 1$.

Expanding $|\lambda I - L_n^c|$ with respect to the first row and using (6) we have:

$$\Delta_L(\lambda) = (\lambda - 1)Z_{n-1}(\lambda) - Z_{n-2}(\lambda) = Z_n(\lambda) + Z_{n-1}(\lambda).$$

By item 2 of Corollary to Lemma 1 the roots of $Z_n(\lambda) + Z_{n-1}(\lambda)$ are the squares of the roots of $P_{2n}(\lambda) + P_{2(n-1)}(\lambda)$.

By (4),

$$P_{2n}(\lambda) + P_{2(n-1)}(\lambda) = \lambda P_{2n-1}(\lambda).$$

According to (5) the roots of $\lambda P_{2n-1}(\lambda)$ are $\lambda_0 = 0$ and $\lambda_k = 2\cos\frac{\pi k}{2n}$, $k = 1, \ldots, 2n-1$. $P_{2n-1}(\lambda)$ has roots 0 (corresponding to $k = n$), $n-1$ positive roots and $n-1$ negative roots with the same absolute values. Since $\Delta_L(\lambda)$ has n roots equal to the squares of the roots of $\lambda P_{2n-1}(\lambda)$. The roots of $\Delta_L(\lambda)$ make up the set $\{4\cos^2\frac{\pi k}{2n}, k = 1, \ldots, n\}$. It is easy to verify that the numbers $4\sin^2\frac{\pi k}{2n}$, $k = 0, \ldots, n-1$, form the same set. This result was proved in [1, 2]. In those papers the statement was taken as "ready-made" and then proved, rather than derived, as we did in this paper.

The matrix tree theorem provides a general formula for the computation of the number of spanning trees of a graph. For certain classes of graphs, the result can be obtained by means of the Chebyshev polynomials of the second kind. In [3] this method was applied to the wheels, fans, Moebius ladders, etc. In [4] was demonstrated the use of Chebyshev polynomials for finding the number of spanning trees in certain classes of graphs such as circulant graphs with fixed jumps and circulant graph with non-fixed jumps. The results obtained in the present paper can be used to find the number of spanning trees for the class of digraphs described above and to solve the problem of the reality of eigenvalues (cf. [5]).

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