ON NONSTATIONARY QUEUEING MODELS WITH CATASTROPHES

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We consider nonstationary Markovian queueing models with catastrophes. The bounds on the rate of convergence to the limit regime and the estimates of the limit characteristics are obtained.

Keywords: Nonstationary queues, Markovian models with catastrophes, weak ergodicity, bounds, limit characteristics.

1. INTRODUCTION

The simplest queueing models with catastrophes have been studied some years ago, see for instance [1, 2, 3, 6]. First results for nonstationary $M(t)/M(t)/S$ queue with catastrophes have been obtained in [10]. Here we consider general nonstationary birth-death process (BDP) with catastrophes.

Our approach is based on the method introduced by Gnedenko and Makarov (see [4]) and successively worked out by one of the authors in [7] and [8].

Let $X = X(t)$, $t \geq 0$ be a BDP with catastrophes, and let $\lambda_n(t)$, $\mu_n(t)$ and $\xi(t)$ be birth, death, and catastrophe rates, respectively.

Let $p_{ij}(s,t) = Pr\left\{X(t) = j\,|X(s) = i\right\}$ for $i, j \geq 0$, $0 \leq s \leq t$ be the transition probability functions of the process $X = X(t)$ and $p_i(t) = Pr\left\{X(t) = i\right\}$ be the state probabilities.

The probabilistic dynamics of the process is represented by the forward Kolmogorov system of differential equations:

$$\left\{ \begin{array}{c} \frac{dp_0}{dt} = -\left(\lambda_0(t) + \xi(t)\right)p_0 + \mu_1(t)p_1 + \xi(t), \\ \frac{dp_k}{dt} = \lambda_{k-1}(t)p_{k-1} - \left(\lambda_k(t) + \mu_k(t) + \xi(t)\right)p_k + \mu_{k+1}(t)p_{k+1}, k \geq 1. \end{array} \right. \tag{1}$$

We denote by $\boldsymbol{p}(t) = (p_0(t), p_1(t), \dots)^T$, $t > 0$ the column vector of state probabilities and by $\boldsymbol{A}(t) = \left\{a_{ij}(t),\ t \geq 0\right\}$ the matrix related to (1).

We shall restrict ourselves to birth and death processes whose rates have the $\lambda_n(t) = \nu_n\lambda(t)$, $\mu_n(t) = \eta_n\mu(t)$, $t \geq 0$, with the assumptions that the rates are bounded, $0 \leq \eta_n \leq M$, $0 \leq \nu_n \leq M$, see [9] for details.

Then we can rewrite the system (1) in the form

$$\frac{dp}{dt} = A(t)p + g(t), \quad t \geq 0, \tag{2}$$

as a differential equation in the space of sequences l_1, where $g(t) = (\xi(t), 0, 0, \dots)^T$.

Throughout the whole paper we assume that $\lambda(t)$, $\mu(t)$ and $\xi(t)$ are locally integrable for $t \geq 0$. Moreover, we suppose (only for simplicity) that these functions are bounded, namely $\lambda(t) + \mu(t) + \xi(t) \leq L < \infty$, for almost all $t \geq 0$.

Then $\|A(t)\|_1 = \sup_j \sum_i |a_{ij}(t)| \leq 2ML$, for almost all $t \geq 0$.

2. BOUNDS, LARGE CATASTROPHE RATE

Theorem 1. *Let*

$$\int_0^\infty \xi(t)\,dt = \infty. \tag{3}$$

Then $X(t)$ is weakly ergodic in uniform operator topology. Moreover, the following bound holds

$$\|p^*(t) - p^{**}(t)\| \leq 2e^{-\int_0^t \xi(\tau)\,d\tau}, \tag{4}$$

for any initial conditions $p^*(0), p^{**}(0)$.

Proof. We use the notion of the logarithmic norm of operator function and the related estimates, see [5] and [9]. We have the equality

$$\gamma(A(t))_1 = \sup_i \left(a_{ii}(t) + \sum_{j \neq i} a_{ji}(t) \right) = -\xi(t). \tag{5}$$

Hence we obtain the following (sharp!) bound:

$$\|U(t,s)\| \leq e^{-\int_s^t \xi(\tau)\,d\tau}, \tag{6}$$

for any $0 \leq s \leq t$, and our claim.

Remark. In the general case we can consider *any* $p^*(t)$ as the limit regime of state probabilities. However, if all intensities ($\lambda(t)$, $\mu(t)$ and $\xi(t)$) are 1-periodic, then there exists 1-periodic limit regime, say $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$.

We shall study the following mean values

$$E_{p(0)}(t) = E_{p(0)}\{X(t)\} = E\{X(t)\,|p(0)\}, \tag{7}$$

and particularly

$$E_k(t) = E\{X(t)\,|X(0) = k\}. \tag{8}$$

Definition. Markov chain $X(t)$ has the *limiting mean* $\varphi(t)$ if

$$\lim_{t\to\infty} (\varphi(t) - E_k(t)) = 0 \tag{9}$$

for any k.

Theorem 2. *Let all birth, death, and catastrophe rates $\lambda(t)$, $\mu(t)$, $\xi(t)$ be 1-periodic. Let*

$$\int_0^1 \xi(t)\,dt > 0. \tag{10}$$

Then X(t) has the 1-periodic limiting mean $\varphi(t)$.

Moreover, the following bound holds:

$$|\varphi(t) - E_0(t)| \le WMe^{-\int_0^t (\xi(\tau) - M(\delta-1)\lambda(\tau))\,d\tau}, \tag{11}$$

where $\delta > 1$ is such that

$$\int_0^1 (\xi(t) - M\,(\delta - 1)\,\lambda(t))\;dt > 0, \tag{12}$$

$$W = \sup_n \frac{n}{\delta^n} < \infty, \tag{13}$$

and

$$M = \limsup_{t\to\infty} \|p(t)\|_B < \infty, \tag{14}$$

where $\|x\|_B = \sum_{i=0}^{\infty} \delta^i |x_i| < \infty$.

Proof follows by the way of the reasoning of Theorem 4 of [10].

3. BOUNDS, GENERAL CASE

Put $p_0(t) = 1 - \sum_{i\ge 1} p_i(t)$ (for ordinary BDP see this way of study, for instance, in [9]), then we obtain the following system from (2)

$$\begin{pmatrix} \frac{dp_1}{dt} \\ \frac{dp_2}{dt} \\ \vdots \\ \frac{dp_n}{dt} \\ \vdots \end{pmatrix} = \begin{pmatrix} -(\lambda_0 + \lambda_1 + \mu_1 + \xi) & (\mu_2 - \lambda_0) & -\lambda_0 & -\lambda_0 & \cdots & \cdots \\ \lambda_1 & -(\lambda_2 + \mu_2 + \xi) & \mu_3 & 0 & 0 & \cdots \\ 0 & \lambda_2 & -(\lambda_3 + \mu_3 + \xi) & \mu_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \tag{15}$$

or otherwise

$$\frac{dz(t)}{dt} = B(t)z(t) + f(t). \tag{16}$$

This is a linear non-homogeneous differential system the solution of which can be written as

$$z(t) = V(t,0)z(0) + \int_0^t V(t,z)f(z)\,dz, \tag{17}$$

where $V(t,z)$ is the Cauchy operator of (16).

Consider the matrix

$$D = \begin{pmatrix} d_0 & d_0 & d_0 & \cdots \\ 0 & d_1 & d_1 & \cdots \\ 0 & 0 & d_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{18}$$

and the space of sequences

$$\ell_{1D} = \{z = (p_1, p_2, \ldots) : \|z\|_{1D} = \|Dz\|_1 < \infty\}, \tag{19}$$

as in [9], where d_i are some positive numbers.

Now we can study BDP with catastrophes using the logarithmic norm and related bounds.

Put

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), \quad k \geq 0, \tag{20}$$

and

$$\alpha(t) = \inf_{k \geq 0} \alpha_k(t). \tag{21}$$

Theorem 3. *Let a process with rates $\lambda_k(t)$, $\mu_k(t)$, and $\xi(t)$ be given.*
Let us assume that there exists a sequence $\{d_j\}$ such that $1 \leq d_1 \leq d_2 \leq \ldots$, and

$$\int_0^\infty \alpha(t)\,dt = +\infty. \tag{22}$$

Then $X(t)$ is weakly ergodic for any $\xi(t)$, and the following bound holds:

$$\|p^*(t) - p^{**}(t)\|_B \leq 2e^{-\int_s^t \alpha(\tau)\,d\tau}\|p^*(s) - p^{**}(s)\|_{1D}, \tag{23}$$

for any s, t, $0 \leq s \leq t$, and for any acceptable initial conditions $p^*(s)$, $p^{**}(s)$.

Proof. We have now the following bound of the logarithmic norm $\gamma(B(t))$ in l_{1D}:

$$\gamma(B)_{1D} = \sup_{i \geq 0}\left(\frac{d_{i+1}}{d_i}\lambda_{i+1}(t) - (\lambda_i(t) + \mu_{i+1}(t) + \xi(t)) + \frac{d_{i-1}}{d_i}\mu_i(t)\right) \leq -\alpha(t), \tag{24}$$

in accordance with (20). Hence, using the reasoning of Theorem 1 of [9], we obtain the inequality

$$\|p^*(t) - p^{**}(t)\|_{1D} \le e^{-\int_s^t \alpha(\tau)\,d\tau} \|p^*(s) - p^{**}(s)\|_{1D}. \tag{25}$$

Consider B and l_{1D} norms of a vector $z = (z_1, z_2, \dots)^T$, then

$$\|z\|_B = \sum_{i\ge 1} d_i z_i = d_1 \left(\left| \sum_{i\ge 1} z_i + \sum_{i\ge 2} -z_i \right| \right) + d_2 \left(\left| \sum_{i\ge 2} z_i + \sum_{i\ge 3} -z_i \right| \right) + \dots \le \tag{26}$$

$$d_1 \left| \sum_{i\ge 1} z_i \right| + 2d_2 \left| \sum_{i\ge 2} z_i \right| + \dots \le 2\|z\|_{1D}, \tag{27}$$

and we obtain our claim.

Corollary. Let, in addition, the numbers d_i grow sufficiently fast so that $\inf_{k\ge 1} \frac{d_k}{k} = \omega > 0$. Then $X(t)$ has the limiting mean, say $\phi^*(t)$, and the following bound holds:

$$|\phi^*(t) - E_k(t)| \le \frac{2}{\omega} e^{-\int_0^t \alpha(\tau)\,d\tau} \|p^*(0) - e_k\|_{1D}. \tag{28}$$

Theorem 4. *Let under assumptions of the previous Corollary all intensities be 1-periodic. Then there exists 1-periodic limit regime, say $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$, and the respective limiting mean $\phi(t)$. Moreover, the following bounds hold:*

$$\|p(t) - \pi(t)\|_B \le 2e^{-\int_0^t \alpha(\tau)\,d\tau} \|p(0) - \pi(0)\|_{1D}, \tag{29}$$

$$|\phi(t) - E_k(t)| \le \frac{2}{\omega} e^{-\int_0^t \alpha(\tau)\,d\tau} \|\pi(0) - e_k\|_{1D}. \tag{30}$$

Remark. We can obtain the bound for $\|\pi(0)\|_{1D}$ using the approach of [9]. We have

$$\sup_{|t-s|\le 1} \int_s^t \alpha(\tau)\,d\tau = K < \infty, \tag{31}$$

and

$$\limsup_{t\to\infty} \|\pi(t)\|_{1D} \le \left\| \int_0^t V(t,\tau) f(\tau) d\tau \right\|_{1D} \le L\nu_0 \int_0^t e^{\int_\tau^t \alpha(u)\,du} d\tau \le$$

$$Le^K \nu_0 \int_0^t e^{-\alpha^*(t-\tau)} d\tau \le \frac{Le^K \nu_0}{\alpha^*}, \tag{32}$$

where $\alpha^* = \int_0^1 \alpha(u)\,du$.

Now, $\|\pi(0)\|_{1D} \leq \limsup_{t\to\infty} \|\pi(t)\|_{1D}$ by 1-periodicity of $\pi(t)$. Hence we obtain the following bound:

$$\|\pi(0) - \boldsymbol{e_k}\|_{1D} \leq \limsup_{t\to\infty} \|\pi(t)\|_{1D} + \|\boldsymbol{e_k}\|_{1D}, \tag{33}$$

and

Theorem 5. *Let under assumptions of the previous Corollary $X(t) = k$. Then the following bounds hold:*

$$\|\boldsymbol{p}(t) - \pi(t)\|_B \leq 2e^{-\int_0^t \alpha(\tau)\,d\tau} \left(\sum_{i=1}^{k} d_i + \frac{Le^K \nu_0}{\alpha^*}\right), \tag{34}$$

and

$$|\phi(t) - E_k(t)| \leq \frac{2}{\omega} e^{-\int_0^t \alpha(\tau)\,d\tau} \left(\sum_{i=1}^{k} d_i + \frac{Le^K \nu_0}{\alpha^*}\right). \tag{35}$$

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