REPLENISHMENT OF A PERISHABLE PRODUCT IN THE CONDITIONS OF FLUCTUATING DEMAND AND FIXED LEAD TIME

A. Babitski

Belarusian State University
Minsk, Belarus
babitski@hotmail.com

An inventory system with perishable products, deterministic shelf life, and replenishment lead time is studied. Demand is represented by Poisson process with different intensities for different time intervals. Formulae for calculation of expected storage cost and income are obtained. Algorithm for determining optimal replenishment order size for maximization of net income is proposed.

Keywords: inventory control optimization, perishable product, stochastic demand, deterministic lead time.

1. INTRODUCTION

The purpose of this paper is to determine optimal replenishment order size in a system with exponentially distributed fluctuating demand where replenishment moments don't coincide with perishable inventory expiration moments. That is when new orders can be added to the existing inventory before it has to be discarded.

Analysis of systems with fluctuating demand typically leads to development of more complex algorithms, comparing to the systems with constant demand parameters, however such algorithms provide significantly more accurate calculation of costs and ordering recommendations in the majority of practically important cases. Many studied systems, e.g., [2, 7], limit variation of demand intensities to the moments of inventory replenishment. A system with deterministic demand and intensities that may change between replenishment moments is studied in [8]. Current paper allows arbitrary fragmentation of the given time interval with arbitrary assignment of demand intensity values to each fragment, what allows to approximate real demand with any preset precision.

Perishable systems are studied in many papers, e.g., [4, 5, 6], with various assumptions about product deterioration. Some papers, [3, 7], consider situations when ordered items can be delivered after some time, what introduces uncertainty about actual inventory level at the moment of order delivery. As an extension of the results obtained in [1], this paper focuses on the cases of fixed order delivery and discard moments, in particular, inventory discard moments of existing and newly delivered items can be different.

This paper provides formulae for calculation of expected storage cost and income in the system. Algorithm for solving optimization tasks for net income maximization is proposed.

2. MODEL

Behavior of a perishable inventory system with r items at moment 0 is studied up to the moment T of discarding of all unsold items. It is assumed that time between inventory consumption moments is exponentially distributed with intensity μ_i which is constant during time interval $[t_i, t_{i+1})$, $0 = t_0 \leq t_i < t_N = T$, $0 \leq i < N$. Items are consumed in FIFO order. Items ordered at moment 0 will be delivered at moment t_L, unsold items that were available at moment 0 have to be discarded at moment t_M, $0 \leq L \leq M \leq N$, while the remaining items can be sold up to moment t_N. Items are sold at price s, items which shelf life expired at moments t_L and t_N are utilized with unit income q. Incurred costs include item price p $(q < p < s)$ and cost h for storing one item per unit time.

Because there is no possibility to interfere in the state of the system before time t_L, income and expenses are calculated for time interval $[t_L, t_N]$. So that, if l items were ordered at moment 0, net income in the system can be expressed as:

$$I(l) = -pl + \sum_{k=0}^{r} \psi_{0,L,r-k}^{(r)} (P(k) - S(k))$$

where $P(k)$ is income resulted from sales or inventory clearance and $S(k)$ is total storage cost during time interval $[t_L, t_N]$, and $\psi_{a,b,j}^{(i)}$ is the probability to sell j items during time interval $[t_a, t_b]$ if inventory level at moment t_a was i:

$$\psi_{a,b,j}^{(i)} = \begin{cases} 0, a = b, j > 0 \\ 1, a = b, j = 0 \\ \left(\sum_{m=a}^{b-1} \mu_m (t_{m+1} - t_m)\right)^j e^{-\sum_{m=a}^{b-1} \mu_m (t_{m+1} - t_m)} / j!, a < b, j < i \\ 1 - \sum_{m=0}^{i-1} \psi_{a,b,m}^{(i)}, a < b, j = i \end{cases}$$

$$P(k) = \sum_{j=0}^{k-1} \psi_{L,M,j}^{(k)} \left(sj + q(k-j) + \sum_{\nu=0}^{l} \psi_{M,N,\nu}^{(l)} (s\nu + q(l - \nu)) \right) +$$

$$+ \sum_{j=k}^{l+k} \psi_{L,M,j}^{(l+k)} \left(sj + \sum_{\nu=0}^{l+k-j} \psi_{M,N,\nu}^{(l+k-j)} (s\nu + q(l + k - j - \nu)) \right)$$

$$S(k) = \sum_{j=0}^{k+l-1} \sum_{i=L}^{M-1} \psi_{L,i,j}^{(k+l)} \phi_{i,l+k-j} (t_{i+1} - t_i) +$$

$$+ \sum_{j=0}^{k-1} \psi_{L,M,j}^{(k)} \sum_{\nu=0}^{l-1} \sum_{i=M}^{N-1} \psi_{M,i,\nu}^{(l)} \phi_{i,l-\nu} (t_{i+1} - t_i) +$$

$$+\sum_{j=k}^{l+k-1} \psi_{L,M,j}^{(l+k)} \sum_{\nu=0}^{l+k-j-1} \sum_{i=M}^{N-1} \psi_{M,i,\nu}^{(l+k-j)} \phi_{i,l+k-j-\nu}(t_{i+1}-t_i)$$

where $\phi_{i,j}(x)$ is expected total storage cost in the system from moment t_i during time x, $t_i + x < t_{i+1}$ if inventory level at moment t_i was j. For positive values of μ_i it can be represented by recursive formulae:

$$\phi_{i,1}(x) = \frac{h\left(1-e^{-\mu_i x}\right)}{\mu_i},$$

$$\phi_{i,j}(x) = hjxe^{-\mu_i x} + \int_0^x \left(hjt + \phi_{i,j-1}(x-t)\right) d\left(1-e^{-\mu_i t}\right), j > 1.$$

Applying Laplace transform $\Phi_{i,j}(\tau) = \int_0^\infty \phi_{i,j}(x) e^{-\tau x} dx$, obtain

$$\Phi_{i,j}(\tau) = \frac{hj\left(\mu_i - \tau\right)}{\tau\left(\mu_i + \tau\right)^2} + \frac{\mu_i}{\mu_i + \tau} \Phi_{i,j-1}(\tau).$$

That leads to direct formula for $\Phi_{i,j}(\tau)$

$$\Phi_{i,j}(\tau) = h \sum_{k=1}^{j} \frac{k\mu_i^{j-k}}{\tau\left(\mu_i + \tau\right)^{j-k+1}}.$$

Now explicit formula for $\phi_{i,j}(x)$ can be found, using inverse transform:

$$\phi_{i,j}(x) = \begin{cases} hjx, \mu_i = 0 \\ \frac{h}{\mu_i} \sum_{k=1}^{j} k\left(1 - \sum_{n=0}^{j-k} \frac{(\mu_i x)^n}{n!} e^{-\mu_i x}\right), \mu_i > 0. \end{cases}$$

3. OPTIMIZATION PROBLEM

Optimization task is formulated as

$$I(l) \to \max_l .$$

Search for the optimal value of l relies on convexity of $I(l)$. As can be seen, increments of $I(l)$

$$\Delta I(l) = I(l+1) - I(l) = -p + \sum_{k=0}^{r} \psi_{0,L,r-k}^{(r)} \left[\sum_{j=0}^{k-1} \psi_{L,M,j}^{(k)} \left(\sum_{\nu=0}^{l} \psi_{M,N,\nu}^{(l+1)} (q-s) + s \right) + \right.$$

$$+ \sum_{j=k}^{l+k} \psi_{L,M,j}^{(l+k+1)} \left(\sum_{\nu=0}^{l+k-j} \psi_{M,N,\nu}^{(l+k+1)} (q-s) + s \right) - \sum_{i=L}^{M-1} \frac{h}{\mu_i} \sum_{j=0}^{l+k} \psi_{L,i,j}^{(l+k+1)} \left(k+l+1-j- \right.$$

$$\left. - \sum_{\tau=0}^{l+k-j} (k+l+1-j-\tau) \frac{\left(\mu_i\left(t_{i+1}-t_i\right)\right)^\tau}{\tau!} e^{-\mu_i(t_{i+1}-t_i)} \right) -$$

$$-\sum_{j=0}^{k-1}\psi_{L,M,j}^{(k)}\sum_{i=M}^{N-1}\frac{h}{\mu_i}\sum_{\nu=0}^{l}\psi_{M,i,\nu}^{(l+1)}(l+1-\nu-$$

$$-\sum_{\tau=0}^{l-\nu}(l+1-\nu-\tau)\frac{(\mu_i\,(t_{i+1}-t_i))^\tau}{\tau!}e^{-\mu_i(t_{i+1}-t_i)}\Bigg)-$$

$$-\sum_{i=M}^{N-1}\frac{h}{\mu_i}\sum_{j=k}^{l+k}\psi_{L,M,j}^{(l+k+1)}\Bigg(\sum_{\nu=0}^{l+k-j}\psi_{M,i,\nu}^{(l+k+1)}(k+l+1-j-\nu-$$

$$-\sum_{\tau=0}^{l+k-j-\nu}(k+l+1-j-\nu-\tau)\frac{(\mu_i\,(t_{i+1}-t_i))^\tau}{\tau!}e^{-\mu_i(t_{i+1}-t_i)}\Bigg)\Bigg)\Bigg]$$

are decreasing monotonically with growth of l, because increments of $\Delta I(l)$ are always negative:

$$\Delta I(l+1)-\Delta I(l)=$$

$$=-\sum_{k=0}^{r}\psi_{0,L,r-k}^{(r)}\Bigg[(s-q)\Bigg(\sum_{j=0}^{k-1}\psi_{L,M,j}^{(k)}\psi_{M,N,l+1}^{(l+2)}+\sum_{j=k}^{k+l+1}\psi_{L,M,j}^{(k+l+2)}\psi_{M,N,l+1+k-j}^{(k+l+2)}\Bigg)+$$

$$+\sum_{i=L}^{M-1}\frac{h}{\mu_i}\sum_{j=0}^{l+k+1}\psi_{L,i,j}^{(l+k+2)}\Bigg(1-\sum_{\tau=0}^{l+k+1-j}\frac{(\mu_i\,(t_{i+1}-t_i))^\tau}{\tau!}e^{-\mu_i(t_{i+1}-t_i)}\Bigg)+$$

$$+\sum_{j=0}^{k-1}\psi_{L,M,j}^{(k)}\sum_{i=M}^{N-1}\frac{h}{\mu_i}\sum_{\nu=0}^{l+1}\psi_{M,i,\nu}^{(l+2)}\Bigg(1-\sum_{\tau=0}^{l+1-\nu}\frac{(\mu_i\,(t_{i+1}-t_i))^\tau}{\tau!}e^{-\mu_i(t_{i+1}-t_i)}\Bigg)+$$

$$+\sum_{i=M}^{N-1}\frac{h}{\mu_i}\sum_{j=k}^{k+l+1}\psi_{L,M,j}^{(l+k+2)}\Bigg(\sum_{\nu=0}^{l+k+1-j}\psi_{M,i,\nu}^{(l+k+2)}(1-\sum_{\tau=0}^{l+1+k-\nu-j}\frac{(\mu_i\,(t_{i+1}-t_i))^\tau}{\tau!}e^{-\mu_i(t_{i+1}-t_i)}\Bigg)\Bigg)\Bigg].$$

So that, value l^* that results in maximal value of net income can be found by binary search. This is the value that satisfies the conditions $\Delta I(l^*-1)>0$, $\Delta I(l^*)<0$.

4. NUMERICAL EXAMPLES

Tables below contain values of $I(l)$ calculated for various assumptions about demand and initial inventory level with the following fixed parameters $L=2$, $M=4$, $N=5$, $t_1=4$, $t_2=6$, $t_3=8$, $t_4=10$, $t_5=12$, $h=2.8$, $p=14$, $q=5$, $s=32$. Columns in table 1 contain values of net income for the set of $(\mu_0,\mu_1,\mu_2,\mu_3,\mu_4)$ specified in the header for values of replenishment quantity l ranging from 1 to 20, initial inventory level is fixed as $r=12$.

Table 1 **Impact of fluctuating demand**

l	(3,3,3,3,3)	(3,3,3,1,5)	(3,1,5,1,5)	(3,5,1,5,1)	(7,1,1,1,1)	(1,7,7,1,1)
1	20.35	20.29	37.33	15.87	15.14	21.06
2	36.37	36.27	53.81	29.46	27.13	38.21

Table 1 **Impact of fluctuating demand (continued)**

l	(3,3,3,3,3)	(3,3,3,1,5)	(3,1,5,1,5)	(3,5,1,5,1)	(7,1,1,1,1)	(1,7,7,1,1)
3	51.47	51.22	69.70	41.78	35.28	54.96
4	65.63	65.05	84.97	53.20	**38.65**	71.31
5	78.85	77.62	99.58	63.89	36.41	87.25
6	91.13	88.85	113.47	73.85	28.27	102.76
7	102.45	98.70	126.54	82.94	14.70	117.81
8	112.78	107.19	138.69	90.88	-3.31	132.31
9	122.05	114.39	149.82	97.26	-24.54	146.15
10	130.16	120.32	159.84	101.55	-47.85	159.10
11	136.92	124.95	168.68	**103.21**	-72.40	170.86
12	142.10	128.15	176.27	101.71	-97.62	181.03
13	145.41	**129.68**	182.55	96.72	-123.17	189.16
14	**146.52**	129.26	187.43	88.08	-148.86	194.76
15	145.09	126.57	190.80	75.89	-174.62	**197.38**
16	140.88	121.33	**192.48**	60.46	-200.41	196.67
17	133.70	113.34	192.29	42.24	-226.20	192.42
18	123.51	102.55	190.02	21.76	-252.00	184.59
19	110.43	89.01	185.46	-0.46	-277.80	173.33
20	94.66	72.92	178.47	-23.94	-303.60	158.93

Values of μ_i were chosen with the purpose to preserve the same average demand intensity on interval $[t_0, t_5]$. It is easy to see that fluctuations in demand intensity significantly affect not only net income in the system but also the optimal replenishment quantity.

Typical practical approach that is used widely in ordering planning with lead time calculates optimal order quantity from optimal order quantity for an empty system by subtracting av-erage consumption of initial inventory during lead time. Table 2 demonstrates that increments of initial inventory don't directly translate to decrements of optimal order quantities. Net income values were calculated for demand intensities ($\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 2$, $\mu_4 = 4$).

Table 2 **Impact of initial inventory level**

l	$r = 0$	$r = 6$	$r = 12$	$r = 18$	$r = 24$
1	16.60	16.60	17.73	40.54	113.47
2	31.80	31.81	32.87	54.55	123.68
3	45.61	45.62	46.63	67.21	132.69
4	58.05	58.06	59.02	78.53	140.42
5	69.15	69.16	70.07	88.52	146.68
6	78.96	78.97	79.83	97.18	151.21
7	87.52	87.53	88.34	104.49	153.68
8	94.85	94.85	95.60	110.38	**153.83**
9	100.88	100.88	101.57	114.73	151.45
10	105.46	105.46	106.07	117.36	146.44

Table 2 **Impact of initial inventory level (continued)**

l	$r=0$	$r=6$	$r=12$	$r=18$	$r=24$
11	108.35	108.35	108.87	**118.01**	138.78
12	**109.21**	**109.22**	**109.63**	116.42	128.54
13	107.68	107.68	107.98	112.31	115.84
14	103.42	103.42	103.60	105.45	100.82
15	96.19	96.19	96.25	95.74	83.70
16	85.90	85.90	85.85	83.19	64.71
17	72.63	72.63	72.48	67.94	44.10
18	56.60	56.60	56.37	50.27	22.14
19	38.15	38.14	37.84	30.52	-0.90
20	17.67	17.67	17.31	9.06	-24.78

REFERENCES

1. *Babitski A.* Optimization of inventory management of a perishable product in the conditions of fluctuating demand // Computer Modelling and New Technologies. 2007. V. 11. № 4. P. 37–40.
2. *Cheng F., Sethi S. P.* Optimality of State-Dependent (s,S) Policies in Inventory Models With Markov-Modulated Demand and Lost Sales // Production and Operations Management. 1999. V. 8(2). P. 183–192.
3. *Kalpakam S., Sapna K. P.* Continuous review (s,S) inventory system with random lead-times and positive lead-times // OR Letters. 1994. V. 16. P. 115–119.
4. *Kanchanasuntorn K., Techanitisawad A.* An approximate periodic model for fixed-life perishable products in a two-echelon inventory-distribution system // International Journal of Production Economics. 2006. V. 100(1). P. 101–115.
5. *Nahmias S.* Perishable inventory theory // Operations Research. 1982. V. 30. P. 680–708.
6. *Pal M.* An inventory model for deteriorating items when demand is random // Calcutta Statistical Association Bulletin. 1990. V. 39. P. 201–207.
7. *Ravichandran N.* Probabilistic analysis of a continuous review perishable system with Markovian demand. Erlangian live and non-instanteneous lead times // OR Spektrum. 1988. V. 10. P. 23–27.
8. *Teng J. T., Chern M.-S., Chang Y.-L.* Deterministic Inventory Lot-size models with shortages for fluctuating demand and unit purchase cost // International Transactions in Operational Research. 2005. V. 12. № 1. P. 83–100.