Application of Mathematics in Engineering and Economics

Proceedings of the 29th International Summer School

June 2003, Sozopol – Bulgaria

Editors: Marin Marinov and Georgi Venkov

Faculty of Applied Mathematics and Informatics
Technical University of Sofia

Bulvest 2000
Sofia, 2004

Decomposition of linear systems in network optimization problems

Ludmila A. Pilipchuk, Yuliana H. Pesheva*, Yulia V. Malahovskaya

*Faculty for Applied Mathematics and Computer Scienc, Belarussian State University, F. Skarina Avenue 4, 220050, Minsk, Belarus, *Faculty for Applied Mathematics and Informatics, Technical University of Sofia, POB 384, Sofia 1000, Bulgaria*

Phones: +375-17-2095079, ++3592-9652350

E-mails: pilipchuk@bsu.by, *ylp@tu-sofia.acad.bg*

Abstract

Algorithms for solving linear systems based on decomposition of restrictions are considered. Transformation of the support set of elements and elements of the working support matrix in algorithms for solving network optimization problems is executed.

Decomposition, optimization, network, support, system, determinant of the structure, matrix

1. Problem definition

Let $G=\{I,U\}$ be a finite oriented graph without multiple arcs and loops. Consider the linear underdetermined system

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} a_i, & i\in I\setminus I^*, \\ x_i sign[i], & i\in I^*, \end{cases} \quad sign[i]=\begin{cases} 1, & i\in I^n, \\ -1, & i\in I^*\setminus I^n, \end{cases} \quad I^n\subseteq I^* \tag{1}$$

$$\sum_{(i,j)\in U} \lambda_{ij}^p x_{ij} = \alpha_p, \quad p=\overline{1,q}, \tag{2}$$

$$I_i^+(U)=\{j:\ (i,j)\in U\}, \qquad I_i^-(U)=\{j:\ (j,i)\in U\}$$

We know from [1, 2] that the rank of the matrix of system (1) for a connected graph $G=\{I,U\}$ is equal to $|I|$.Consider $R=\{U_R, I_R^*\}$ - a support of the graph $G=\{I,U\}$ for system (1) [3,4]. Let us define a characteristic vector $\delta_{\tau\rho}=\left(\delta_{ij}^{\tau\rho},(i,j)\in U;\delta_i^{\tau\rho}, i\in I^*\right)$ entailed by an arc $(\tau,\rho)\in U\setminus U_R$ and a characteristic vector $\delta_\gamma=\left(\delta_{ij}^{\gamma},(i,j)\in U;\delta_i^{\gamma}, i\in I^*\right)$ entailed by a node $\gamma\in I^*\setminus I_R^*$ [3,4].

2. Decomposition of the system

<u>Theorem 1.</u> *The general solution of system (1) may be uniquely represented using the following look:*

$$x_{ij} = \sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho}\delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \delta_{ij}^{\gamma} + \tilde{x}_{ij},\ (i,j)\in U_R \tag{3}$$

$$x_i = \sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho}\delta_i^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \delta_i^{\gamma} + \tilde{x}_i,\ i\in I_R^*\cap I(U_T^k),\ \ k=\overline{1,s}, \tag{4}$$

where $\tilde{x}=(\tilde{x}_{ij},\ (i,j)\in U,\ \tilde{x}_i,\ i\in I^*)$ *is a partial solution of the nonhomogeneous system (1).*

Let $R=\{U_R, I_R^*\}$ be a support of the graph $G=\{I,U\}$ for system (1). In arbitrary order, we choose sets $W=\{U_W, I_W^*\}$, $|W|=q$, $U_W \subseteq U \setminus U_R$, $I_W^* \subseteq I^* \setminus I_R^*$. By substituting the general solution of system (1), which has the form (3)-(4) into the system of linear equations (2), we obtain:

$$\sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho}\left[\sum_{(i,j)\in U_R}\lambda_{ij}^p\delta_{ij}^{\tau\rho}+\lambda_{\tau\rho}^p\right]+\sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \sum_{(i,j)\in U_R}\lambda_{ij}^p\delta_{ij}^\gamma+\sum_{(i,j)\in U_R}\lambda_{ij}^p\tilde{x}_{ij}=\alpha_p, \quad (5)$$

The number $\Lambda_{\tau\rho}^p=\sum_{(i,j)\in U_R}\lambda_{ij}^p\delta_{ij}^{\tau\rho}+\lambda_{\tau\rho}^p$ is called the **determinant of the structure entailed by arc** $(\tau,\rho)\in U\setminus U_R$ relatively to restriction (2) with the number p. The number $\Lambda_\gamma^p=\sum_{(i,j)\in U_R}\lambda_{ij}^p\delta_{ij}^\gamma$ *is called the* ***determinant of the structure entailed by node*** $\gamma\in I^*\setminus I_R^*$ *relatively to restriction (2) with number* p.

System (2) takes the form

$$\sum_{(\tau,\rho)\in U_W}\Lambda_{\tau\rho}^p x_{\tau\rho}+\sum_{\gamma\in I_W^*}\Lambda_\gamma^p x_\gamma=\mathrm{A}^p-\sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)}\Lambda_{\tau\rho}^p x_{\tau\rho}-\sum_{\gamma\in I^*\setminus(I_W^*\cup I_R^*)}\Lambda_\gamma^p x_\gamma,\ p=\overline{1,q} \quad (6)$$

$$\mathrm{A}^p=\alpha_p-\sum_{(i,j)\in U_R}\lambda_{ij}^p\tilde{x}_{ij}$$

3. Network support criterion

Theorem 2. (Network Support Criterion) *The aggregate of sets* $K=\{U_K, I_K^*\}$ *is a support of the network* $G=\{I,U\}$ *for system (1)-(2) if and only if*

1) the aggregate of sets $K=\{U_K, I_K^*\}$ *may be divided into two aggregates:* $R=\{U_R, I_R^*\}$ *and* $W=\{U_W, I_W^*\}$, *such as* $U_R\cup U_W=U_K$, $U_R\cap U_W=\varnothing$, $I_R^*\cup I_W^*=I_K^*$, *and set* R *is a support of the network* $G=\{I,U\}$ *for system (1);*

2) $|W|=q$, *where* q *is the number of linearly independent equations in system (2);*

3) matrix D, *which consists of determinants of the structures entailed by the arcs and nodes of aggregate* W, *is nondegenerate.*

4. Support substitution

The final step in solving optimization problems using the support technique is the support substitution. Consider the following cases.

4.1 Introduction of a new arc into the support structure

Let the new element be an arc (i_0, j_0). We call support $K=(U_K, I_K^*)$ of system (1)-(2) *an old support*, and support $\tilde{K}=(\tilde{U}_K, \tilde{I}_K^*)$ - *a new support* of system (1)-(2). We shall mark with a wave all variables related to the new support.

Consider the decomposition of a column of the matrix of system (1), which corresponds to the new arc, on columns of the old support. The coefficients of this decomposition have the following look

$$\mu_{ij} = \sum_{(\tau,\rho)\in U_W} \mu_{\tau\rho}\delta_{ij}^{\tau\rho} + \sum_{\gamma\in I_W^*} \mu_\gamma \delta_{ij}^\gamma - \delta_{ij}^{i_0 j_0},\ (i,j)\in U_R,$$

$$\mu_i = \sum_{(\tau,\rho)\in U_W} \mu_{\tau\rho}\delta_i^{\tau\rho} + \sum_{\gamma\in I_W^*} \mu_\gamma \delta_i^\gamma - \delta_i^{i_0 j_0},\ i\in I_R^*.$$

$$\mu_W = D^{-1}M,\ M=(M^1,...,M^q),\ M^p = \lambda_{i_0 j_0}^p + \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{i_0, j_0},\ p=\overline{1,q}.$$

It is obvious, that non-zero coefficients in decomposition of the column of the matrix of System (1), which corresponds to the new arc, can be obtained only on those arcs and nodes, which are included into structures, entailed by elements of the aggregate W and the arc (i_0, j_0). It is known that in the support substitution only an element with a non-zero coefficient in the decomposition of a new column on the old support can be removed.

Let an arc (i_0, j_0) be a new support element. We use the following definitions: $L(i,j)$ - a cycle entailed by arc (i,j); $C(i,j)$ - a chain entailed by arc (i,j); $C(\gamma)$ - a chain entailed by node γ.

Consider the following cases:

4.1.1. We delete an arc from the support tree. Let the arc we delete belongs to the support tree: $(i_*, j_*)\in U_R$. Let $T=\{I(U_T), U_T\}$ be a subtree with the root $k=\arg\max\{level(i_*), level(j_*)\}$. The next cases can occur:

1). If $(i_*, j_*)\in L(i_0, j_0)$, then the next determinants will be changed:

$$\widetilde{\Lambda}_{\tau\rho}^p = \Lambda_{\tau\rho}^p + sign(i_0, j_0)^{\widetilde{L}(\tau,\rho)} \cdot \Lambda_{i_0 j_0}^p,\quad p=\overline{1,q},\quad \text{for all}\quad (\tau,\rho)\in U_W,$$
$$\tau\in I(U_T)\wedge\rho\notin I(U_T)\vee\tau\notin I(U_T)\wedge\rho\in I(U_T).$$

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p + sign(i_0, j_0)^{\widetilde{C}(\gamma)} sign(\gamma)\cdot\Lambda_{i_0 j_0}^p,\ p=\overline{1,q},\ \text{for } \gamma\in I_W^*\cap I(U_T).$$

2). If $(i_*, j_*)\notin L(i_0, j_0)$, but $\exists (i_1, j_1)\in U_W$, and $(i_*, j_*)\in L(i_1, j_1)$, then we'll get one of the following:

a) If $\exists k\in[1,s]$, $\{i_0, j_0\}\subset I(U_T^k)$, then the next set of changes will take place:

$$\widetilde{\Lambda}_{\tau\rho}^p = \Lambda_{\tau\rho}^p + sign(i_1, j_1)^{\widetilde{L}(\tau,\rho)} \cdot \Lambda_{i_1 j_1}^p,\quad p=\overline{1,q},\quad \text{for all}\quad (\tau,\rho)\in U_W,$$
$$\tau\in I(U_T)\wedge\rho\notin I(U_T)\vee\tau\notin I(U_T)\wedge\rho\in I(U_T).$$

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p + sign(i_1, j_1)^{\widetilde{C}(\gamma)} sign(\gamma)\cdot\Lambda_{i_1 j_1}^p,$$

In Matrix D elements $\Lambda_{i_1 j_1}^p, p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0 j_0}^p, p=\overline{1,q}$.

b) If $\exists k,l\in[1,s], k\neq l,\ i_0\in I(U_T^k),\ j_0\in I(U_T^l)$, then

$$\widetilde{\Lambda}_{\tau\rho}^p = \Lambda_{\tau\rho}^p + sign(i_1, j_1)^{\widetilde{L}(\tau,\rho)} \cdot \Lambda_{i_1 j_1}^p,\quad p=\overline{1,q},\quad \text{for all}\quad (\tau,\rho)\in U_W,$$
$$\tau\in I(U_T)\wedge\rho\notin I(U_T)\vee\tau\notin I(U_T)\wedge\rho\in I(U_T).$$

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p + sign(i_1, j_1)^{\widetilde{C}(\gamma)} sign(\gamma)\cdot\Lambda_{i_1 j_1}^p,\ p=\overline{1,q},\ \text{for } \gamma\in I_W^*\cap I(U_T).$$

$\Lambda_{i_1 j_1}^p, p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_u^p, p=\overline{1,q}$, where $u=I_R^*\cap I(U_T^h)$, $h=k\vee h=l$.

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(u) \cdot \widetilde{\Lambda}_{u}^{p}$, $p = \overline{1,q}$, for $\gamma \in I_W^* \cap I(U_T^h)$.

3). If $\nexists (i,j) \in U_W$, $(i_*, j_*) \in L(i,j)$, but $(i_*, j_*) \in C(i_0, j_0)$, then

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(i_0, j_0)^{\widetilde{C}(\gamma)} \Lambda_{i_0 j_0}^{p}$, $p = \overline{1,q}$, $\gamma \in I_W^* \cap I(U_T)$.

4). If $\nexists (i,j) \in U_W$, $(i_*, j_*) \in L(i,j)$ and $(i_*, j_*) \notin C(i_0, j_0)$, but $\exists u \in I_W^*$, $(i_*, j_*) \in C(u)$, then there are three cases:

a) If $\exists k \in [1,s]$, $\{i_0, j_0\} \subset I(U_T^k)$

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} - sign(u) \cdot sign(\gamma) \cdot \Lambda_{u}^{p}$, $p = \overline{1,q}$, for $\gamma \in I_W^* \cap I(U_T) \setminus \{u\}$.

Elements Λ_{u}^{p}, $p = \overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0 j_0}^{p}$, $p = \overline{1,q}$.

b)If $i_0, j_0 \notin I(U_T)$, but $\exists k, l \in [1,s], k \neq l$, $i_0 \in I(U_T^k)$, $j_0 \in I(U_T^l)$, then

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} - sign(u) \cdot sign(\gamma) \cdot \Lambda_{u}^{p}$, $p = \overline{1,q}$, for $\gamma \in I_W^* \cap I(U_T) \setminus \{u\}$.

Elements Λ_{u}^{p}, $p = \overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{v}^{p}$, $p = \overline{1,q}$, where $v = I_R^* \cap I(U_T^h)$, $h = k \vee h = l$.

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(v) \cdot \widetilde{\Lambda}_{v}^{p}$, $p = \overline{1,q}$, $\gamma \in I_W^* \cap I(U_T^h)$.

c) If $i_0 \in I(U_T) \wedge j_0 \notin I(U_T) \vee i_0 \notin I(U_T) \wedge j_0 \in I(U_T)$, then

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(i_0, j_0)^{\widetilde{C}(\gamma)} \cdot \Lambda_{i_0 j_0}^{p}$, $p = \overline{1,q}$, $\gamma \in I_W^* \cap I(U_T)$.

4.1.2. We delete an arc from the auxiliary support set W. Consider $(i_*, j_*) \in U_W$.

1). If $\exists k \in [1,s]$, $\{i_0, j_0\} \subset I(U_T^k)$, then elements $\Lambda_{i_* j_*}^{p}$, $p = \overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0 j_0}^{p}$, $p = \overline{1,q}$ in Matrix D.

2). If $\exists k, l \in [1,s], k \neq l$, $i_0 \in I(U_T^k)$, $j_0 \in I(U_T^l)$, then elements $\Lambda_{i_* j_*}^{p}$, $p = \overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{u}^{p}$, $p = \overline{1,q}$, where $u = I_R^* \cap I(U_T^h)$, $h = k \vee h = l$.

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(u) \cdot \widetilde{\Lambda}_{u}^{p}$, $p = \overline{1,q}$, for all $\gamma \in I_W^* \cap I(U_T^h)$.

4.1.3. We delete a root from the support tree. Consider a case when we delete a node $u = I_R^* \cap I(U_T)$, where $T = \{I(U_T), U_T\}$ is one of the support trees. The next cases may take place:

1). If $i_0 \in I(U_T) \wedge j_0 \notin I(U_T) \vee i_0 \notin I(U_T) \wedge j_0 \in I(U_T)$, then

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} + sign(\gamma) \cdot sign(i_0, j_0)^{\widetilde{C}(\gamma)} \cdot \Lambda_{i_0 j_0}^{p}$, $p = \overline{1,q}$, $\gamma \in I_W^* \cap I(U_T)$.

2). If $i_0, j_0 \notin I(U_T)$, but $\exists k, l \in [1,s], k \neq l$, $i_0 \in I(U_T^k)$, $j_0 \in I(U_T^l)$, then $\exists y \in I_W^* \cap I(U_T)$

$\widetilde{\Lambda}_{\gamma}^{p} = \Lambda_{\gamma}^{p} - sign(\gamma) \cdot sign(y) \cdot \Lambda_{y}^{p}$, $p = \overline{1,q}$, $\gamma \in I_W^* \cap I(U_T) \setminus \{y\}$.

$$\widetilde{\Lambda}^p_\gamma = \Lambda^p_\gamma + sign(\gamma)\cdot sign(i_0, j_0)^{\widetilde{C}(\gamma)}\cdot \Lambda^p_{i_0 j_0}, \qquad p=\overline{1,q}, \qquad \gamma\in I^*_W \cap I(U^h_T),$$
$h=k \vee h=l$.

Elements $\Lambda^p_y, p=\overline{1,q}$ will be replaced by $\widetilde{\Lambda}^p_w, p=\overline{1,q}$, where $w=I^*_R \cap I(U^h_T)$.

3). If $\exists k\in[1,s]$, $\{i_0, j_0\}\subset I(U^k_T)$, then $\exists y\in I^*_W \cap I(U_T)$

$\widetilde{\Lambda}^p_\gamma = \Lambda^p_\gamma - sign(\gamma)\cdot sign(y)\cdot \Lambda^p_y$, $p=\overline{1,q}$, for all $\gamma\in I^*_W \cap I(U_T)\setminus\{y\}$.

Elements $\Lambda^p_y, p=\overline{1,q}$ will be replaced by $\widetilde{\Lambda}^p_{i_0 j_0}, p=\overline{1,q}$.

4.1.4. We delete a node from the auxiliary support set. Consider a case when we delete a node $y\in I^*_W \cap I(U_T)$, where $T=\{I(U_T),U_T\}$ is a tree in the old support. The next cases can occur:

1). If $\exists k\in[1,s]$, $\{i_0, j_0\}\subset I(U^k_T)$, then

elements $\Lambda^p_y, p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}^p_{i_0 j_0}, p=\overline{1,q}$ in Matrix D.

2). If $\exists k,l\in[1,s], k\neq l$, $i_0\in I(U^k_T)$, $j_0\in I(U^l_T)$, then

$\widetilde{\Lambda}^p_\gamma = \Lambda^p_\gamma + sign(\gamma)\cdot sign(i_0, j_0)^{\widetilde{C}(\gamma)}\cdot \Lambda^p_{i_0 j_0}$, $p=\overline{1,q}$, for all $\gamma\in I^*_W \cap I(U^h_T)\setminus\{y\}$, $h=k \vee h=l$.

$\Lambda^p_y, p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}^p_u, p=\overline{1,q}$.

4.2 Introduction of a new node into the support structure

Let i_0 be a new element. Consider the decomposition of a new column in the matrix of system (2) on the columns of the old support. We can see as in the previous case that non-zero coefficients can be obtained only on those arcs and nodes, which are included into structures, entailed by elements of the aggregate W and node i_0.

Let node i_0 be a new support element.

4.2.1. We delete a tree arc.

We delete an arc $\exists(i_*, j_*)\in U_R$. Let $T=\{I(U_T),U_T\}$ be a subtree with a root $k=\arg\max\{level(i_*),level(j_*)\}$. The next cases can occur:

1). If $\exists(i_1, j_1)\in U_W$, $(i_*, j_*)\in L(i_1, j_1)$, then:

$$\widetilde{\Lambda}^p_{\tau\rho} = \Lambda^p_{\tau\rho} + sign(i_1, j_1)^{\widetilde{L}(\tau,\rho)}\cdot \Lambda^p_{i_1 j_1}, \qquad p=\overline{1,q}, \qquad \text{for all} \quad (\tau,\rho)\in U_W,$$
$\tau\in I(U_T)\wedge \rho\notin I(U_T) \vee \tau\notin I(U_T)\wedge \rho\in I(U_T)$.

$\widetilde{\Lambda}^p_\gamma = \Lambda^p_\gamma + sign(i_1, j_1)^{\widetilde{C}(\gamma)} sign(\gamma)\cdot \Lambda^p_{i_1 j_1}$, $p=\overline{1,q}$, for $\gamma\in I^*_W \cap I(U_T)$.

$\Lambda^p_{i_1 j_1}$, $p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}^p_{i_0}$, $p=\overline{1,q}$.

2). If $\nexists(i, j)\in U_W$, $(i_*, j_*)\in L(i, j)$, but $(i_*, j_*)\in C(i_0)$, then

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p - sign(i_0)\cdot sign(\gamma)\cdot \Lambda_{i_0}^p,\ p=\overline{1,q}, \text{ for all } \gamma \in I_W^* \cap I(U_T).$$

3). If $\nexists(i,j)\in U_W$, $(i_*,j_*)\in L(i,j)$, but $\exists i_1 \in I_W^*$, $(i_*,j_*)\in C(i_1)$, then

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p - sign(i_1)\cdot sign(\gamma)\cdot \Lambda_{i_1}^p,\ p=\overline{1,q}, \text{ for all } \gamma \in I_W^* \cap I(U_T)\setminus\{i_1\}.$$

$\Lambda_{i_1}^p$, $p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0}^p$, $p=\overline{1,q}$.

4.2.2. We delete an arc from an auxiliary support set. Let an arc $(i_*,j_*)\in U_W$ be the arc we delete, then elements $\Lambda_{i_*j_*}^p$, $p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0}^p$, $p=\overline{1,q}$.

4.2.3. We delete a root from a support tree. If we delete node $u = I_R^* \cap I(U_T)$ where $T=\{I(U_T),U_T\}$ is one of the support trees, then the following cases can occur:

1). If $i_0 \in I(U_T)$, then

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p - sign(i_0)\cdot sign(\gamma)\cdot \Lambda_{i_0}^p,\ p=\overline{1,q}, \text{ for all } \gamma \in I_W^* \cap I(U_T).$$

2). If $\exists k=[1,s]$, $i_0\in I(U_T^k)$, $T^k \neq T$, then $\exists i_1 \in I_W^* \cap I(U_T)$ and the following elements of Matrix D will be changed:

$$\widetilde{\Lambda}_\gamma^p = \Lambda_\gamma^p - sign(i_1)\cdot sign(\gamma)\cdot \Lambda_{i_1}^p,\ p=\overline{1,q}, \text{ for all } \gamma \in I_W^* \cap I(U_T)\setminus\{i_1\}.$$

$\Lambda_{i_1}^p$, $p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0}^p$, $p=\overline{1,q}$.

4.2.4. We delete a node from an auxiliary support set. If we delete node $y\in I_W^*$ and insert node $i_0\in I^*\setminus(I_R^*\cup I_W^*)$, then one column in matrix D will be changed: elements Λ_y^p, $p=\overline{1,q}$ will be replaced with $\widetilde{\Lambda}_{i_0}^p$, $p=\overline{1,q}$.

Thus, the changes in matrix D can be made by two simple operations: the replacement of one its column by another and the addition to a column the elements of another column multiplied by a scalar.

Let us know matrix $D^{-1}=\{v_{ij}^1,\ i=\overline{1,q},\ j=\overline{1,q}\}$ - the reverse matrix for an unknown matrix $D=\{\mu_{ij},\ i=\overline{1,q},\ j=\overline{1,q}\}$. Let column k of matrix D is changed, and we know the new elements $\widetilde{\mu}_{ik}$, $i=\overline{1,q}$, then the elements of the new reverse matrix $\widetilde{D}^{-1}$ can be found in the following way:

$$\widetilde{v}_{ij} = \sum_{s=1}^{q} t_{is}v_{sj} = \sum_{\substack{s=1\\ s\neq k}}^{q} t_{is}v_{sj} + t_{ik}v_{kj} = \begin{cases} v_{ij} - \frac{f_{ik}}{f_{kk}}v_{kj}, & i\neq k, \\ \frac{1}{f_{kk}}v_{kj}, & i=k, \end{cases} \quad i=\overline{1,q},\ j=\overline{1,q},$$

$$f_{ik} = \sum_{s=1}^{q} v_{is}\widetilde{\mu}_{sk},\ i=\overline{1,q}.$$

Let us add a column k, multiplied by $a\neq 0$ to column *l*, *l#k* Then the new elements of the reverse matrix are:

$$\tilde{v}_{ij} = \sum_{s=1}^{q} t_{is} v_{sj} = \sum_{\substack{s=1 \\ s \neq l}}^{q} t_{is} v_{sj} + t_{il} v_{lj} = \begin{cases} v_{ij}, & i \neq k, i \neq l, \\ v_{kj} - a v_{lj}, & i = k, \\ v_{lj}, & i = l, \end{cases} \qquad i = \overline{1,q},$$

$j = \overline{1,q}$.

References

1. Alsevich V. V., Gabasov R., Glushenkov V. S.: *Linear economic models optimization: Static problems. Teaching aid.* Minsk, BSU, 2000. (In Russian)
2. Gabasov R., Kirillova F. M.: *Linear programming methods. P.3. Special problems.* Minsk, BSU, 1980. (In Russian)
3. Pilipchuk L. A: Solving of linear sparse systems in network optimization problems. *Proceedings of International conference The 3rd scientific readings on ordinary differential equations.* Minsk, 2001. (In Russian)
4. Pilipchuk L. A: Network optimization problems. *Applications of Mathematics in Engineering and Economics '27, eds. D. Ivanchev and M.D. Todorov*, Heron Press, Sofia, 2002.