Proceedings of
the XXVII Summer School, Sozopol'01

Applications of Mathematics in Engineering and Economics

edited by
D. Ivanchev and M.D. Todorov

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TECHNICAL UNIVERSITY OF SOFIA**

HERON PRESS · Sofia · 2002

Applications of Mathematics in Engineering and Economics'27
eds. D. Ivanchev and M.D. Todorov, Heron Press, Sofia, 2002

Network optimization problems

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Let us consider m industrial parts where it is necessary to produce n different products (kinds of works) of a given assortment. Assortment set consists of k_1, k_2,..., k_n product items of kind (1), (2),..., (n) accordingly. The productivity of every part is known: if at the i-th part ($i = \overline{1,m}$) a j-th product ($j = \overline{1,n}$) is produced, then b_{ij} units of this product are produced in a time unit. The problem consists of distributing the works so, that at the time unit maximum quantity of the full assortment sets of products are produced. For the first time this problem was considered in [1].

Denote a part of working time of i-th industrial part, during which j-th product is made by h_{ij} ($i = \overline{1,m}, j = \overline{1,n}$). The design of the optimal plan is reduced to the next problem: let a set of non-negative numbers be given

$$\{b_{ij}\} \quad (i = \overline{1,m},\ j = \overline{1,n}), \quad k_j > 0,\ j = \overline{1,n}$$

where $\max_{1 \le i \le m}\{b_{ij}\} > 0$ ($j = \overline{1,n}$) (*i.e.*, every product must be produced at least at one industrial part). It is necessary to define a set of numbers (plan) $\pi = \{h_{ij}\}, i = \overline{1,m}, j = \overline{1,n}$, proceeding from conditions:

$$h_{ij} \ge 0 \quad (i = \overline{1,m},\ j = \overline{1,n}),$$

$$\sum_{j=1}^{n} h_{ij} \le 1 \quad (i = \overline{1,m}),$$

$$\min_{1 \le j \le n} \frac{\sum_{i=1}^{m} b_{ij} h_{ij}}{k_j} \to \max .$$

That problem is reduced [1] to an industrial-transport problem with additional constraints on the generalized net $S = \{I, U\}$ of the following type:

$$-h_{i_0}/n \to \min,$$

$$\sum_{j\in I_i^+(U)} h_{ij} - \sum_{j\in I_i^-(U)} a_{ji}h_{ji} = \begin{cases} h_i, & i \in I^1, \\ -h_i, & i \in I^2; \end{cases}$$

$$h_{ii_0} = h_{i_0}/n, \quad i \in I_{i_0}^-(U), \quad n = |I^2| - 1,$$

$$h_{ij} \geq 0, \quad (i,j) \in U, \quad 0 \leq h_i \leq 1, \quad i \in I^1, \quad h_i \geq 0, \quad i \in I^2$$

where $I = I^1 \cup I^2$, $I^1 \cap I^2 = \oslash$, $I^1 = \{1, 2, .., m\}$, $I^2 = \{m+1, .., m+n, m+n+1\}$, $i_0 = m+n+1$, $U = \{(i,j), i \in I^1, j \in I^2 \setminus i_0\} \cup \{(i, i_0), i \in I^2 \setminus i_0\}$, $I_i^+(U) = \{j : (i,j) \in U\}$, $I_i^-(U) = \{j : (j,i) \in U\}$.

Let us consider the model of the p-grocery dynamic manufacturing - transport problem on the finite oriented multinetwork $S(t) = \{I, U(t)\}$, $t = \overline{0, T-1}$, varying in some time segment $[0, T-1]$, with set of nodes I, ($n = |I|$) and with set of arcs $U(t)$ ($m(t) = |U(t)|$, O and T are the lists of initial and final nodes):

$$\sum_{t=0}^{T-1}\sum_{q=1}^{p}\Bigg[\sum_{k=1}^{m(t)} c_{k,q}(t) f_{k,q}(t) + \sum_{i=1}^{n}\Big(\hat{c}_{i,q}\,(t+1)\,\hat{x}_{i,q}\,(t+1) + h_{i,q}(t)x_{i,q}(t) + h_{-i,q}(t)x_{-i,q}(t)\Big)\Bigg] \to \min$$

$$\hat{x}_{i,q}\,(t+1) = \hat{x}_{i,q}\,(t) + \sum_{i=T(k)} f_{k,q}(t) - \sum_{i=O(k)} f_{k,q}(t) - x_{i,q}(t) + x_{-i,q}(t) - a_{i,q}(t),$$

$$\sum_{q=1}^{p} \hat{x}_{i,q}\,(t+1) \leq s_i(t+1), \qquad \sum_{q=1}^{p} x_{i,q}(t) \leq \delta_i(t),$$

$$\sum_{q=1}^{p} x_{-i,q}(t) \leq \delta_{-i}(t), \qquad \sum_{q=1}^{p} f_{k,q}(t) \leq d_k(t),$$

$$0 \leq x_{i,q}(t) \leq \delta_{i,q}(t),\ 0 \leq x_{-i,q}(t) \leq \delta_{-i,q}(t),\ 0 \leq \hat{x}_{i,q}\,(t+1) \leq s_{i,q}(t+1),$$

$$0 \leq f_{k,q}(t) \leq d_{k,q}(t),\ \hat{x}_{i,q}\,(0) = b_{i,q} \geq 0,$$

$$t = \overline{0, T-1},\ i = \overline{1, n},\ q = \overline{1, p},\ k = \overline{1, m}$$

where $c_{k,q}(t)$ – transportation cost of a unit of the product q along the arc $k = k(i,j)$ at moment t; $\hat{c}_{i,q}\,(t)$ – cost of storage a unit of the product q in storage i at moment t; $h_{i,q}(t)$ ($h_{-i,q}(t)$) – cost of production (consumption) a unit of the product q in node i at moment t; $f_{k,q}$ – flow of product a along the arc $k = k(i,j)$; $\hat{x}_{i,q}(t)$ – accumulate of the product q in storage i at moment t; $x_{i,q}(t)$ $(x_{-i,q}(t))$ – the amount of units of product q produced (consumed) in node i at moment t; $a_{i,q}(t)$ – constant intensity of product q for node i at moment t ($a_{i,q}(t) > 0$) – item of production, $a_{i,q}(t) < 0$ – item of consumption, $a_{i,q}(t) = 0$ – transit item); $d_k(t)$ – capacity of arc $k = k(i,j)$ at moment t; $d_{k,q}(t)$ – capacity of transportation of product q along arc $k = k(i,j)$ at moment t; $s_i(t)$ – maximum size of storage i at moment t; $s_{i,q}(t)$ – greatest possible amount of product q in storage i at time moment t; $b_{i,q}$ – accumulate of the product in storage i in an initial moment (usually 0); $\delta_i(t)$ $(\delta_{-i}(t))$ – greatest possible amount of production (consumption) of the product in node i at moment t; $\delta_{i,q}(t)$ $(\delta_{-i,q}(t))$ – greatest possible amount of production (consumption) of product q in node i at moment t.

The storages are filled up at the end of period t, and are devastated by the beginning of $t+1$. If it is required to consume all products completely in the indicated period of time, it is necessary to put $\hat{x}_{i,q}(T) = 0$.

Let us assign each site a unique number, defined for each moment t: $i(t) = n \cdot t + i$, $i = \overline{1,n}$, $t = \overline{0,T}$. Set I will increase $(T+1)$ times, as a result we obtain I'. Accordingly will vary the sets $U(t)$. Thus time dependence is stored in clusters, therefore it is possible to consider time independence on arcs. We enumerate arcs: $k(t) = (i(t), j(t)) = (nt+i, nt+j) = \sum_{\tau=0}^{t} m(\tau) + k$, $t = \overline{0,T-1}$, $k = \overline{1,m}$. $k(t)$ will be defined uniquely for each arc. Let us enter supplementary arcs $(i(t), i(t+1))$, which represent the dependence of adjacent periods of time. Let us enumerate supplementary arcs as follows: $\mu + n \cdot t + i$ where

$$\mu = \sum_{t=0}^{T-1} m(t) = \left| \bigcup_{t=0}^{T-1} U(t) \right|, \qquad i = \overline{1,n},\ t = \overline{0,T-1}.$$

As a result we get a new set of arcs, we shall designate it U. Lists of initial and final clusters – O and T accordingly will also change. If $\hat{f}_{i,q}(T) = 0$, it is possible to skip the moment T, for this reason set I should only be increased T times, instead of $(T+1)$ times. Let us perform a variable substitution:

$$\hat{x}_{i,q}(t+1) = f_{(i(t),i(t+1)),q} = f_{\mu+nt+i,q}, \quad t = \overline{0,T-1},\ q = \overline{1,p},$$
$$a_{i,q} = a_{i,q}(0) + b_{i,q}, \quad i = \overline{1,n},\ q = \overline{1,p},$$
$$a_{i,q}(t) = a_{(i(t),i(t+1)),q} = a_{\mu+nt+i,q},$$
$$s_{i,q}(t+1) = d_{(i(t),i(t+1)),q} = d_{\mu+nt+i,q},$$
$$\delta_{i,q}(t) = \delta_{nt+i,q},\ x_{i,q}(t) = x_{nt+i,q}, \quad t = \overline{1,T-1},\ i = \overline{1,n},\ q = \overline{1,p},$$
$$s_i(t+1) = d_{(i(t),\,i(t+1))} = d_{\mu+nt+i}, \quad \delta_i(t) = \delta_{nt+i}, \quad x_i(t) = x_{nt+i},$$
$$t = \overline{1,T-1},\ i = \overline{1,n}.$$

For the matter of convenience let's reassign $n := n \cdot (T+1) = |I'|$.

So, the task (1) is reduced to following non dynamic multigrocery manufacturing - transport problem on the finite oriented multinetwork $R = \{I', U\}$:

$$\sum_{q=1}^{p}\Big(\sum_{k=1}^{m} c_{k,q} f_{k,q} + \sum_{i=1}^{n}\Big(h_{i,q}x_{i,q} + h_{-i,q}x_{-i,q}\Big)\Big) \to \min$$

$$\sum_{i=O(k)} f_{k,q} - \sum_{i=T(k)} f_{k,q} = a_{i,q} + x_{i,q} - x_{-i,q}, \quad i = \overline{1,n},\ q = \overline{1,p},$$

$$\sum_{q=1}^{p} x_{i,q} \le \delta_i, \quad \sum_{q=1}^{p} x_{-i,q} \le \delta_{-i}, \quad \sum_{q=1}^{p} f_{k,q} \le d_k, \quad k = \overline{1,m},$$

$$0 \le x_{i,q} \le \delta_{i,q},\ 0 \le x_{-i,q} \le \delta_{-i,q},\ 0 \le f_{k,q} \le d_{k,q}$$

where $c_{k,q}$ – cost of transportation a unit of product q along arc $k = k(i,j)$; $h_{i,q}$ ($h_{-i,q}$) – cost of production (consumption) a units of product q in node i; $f_{k,q}$ – flow of the product q along the arc $k = k(i,j)$; $x_{i,q}$ ($x_{-i,q}$) – amount of a unit of product q produced (consumed) in node i; $a_{i,q}$ – constant intensity of product q in node i; d_k – capacity of the arc $k = k(i,j)$; $d_{k,q}$ – capacity of transportation of product q along arc $k = k(i,j)$; $\delta_{i,q}$ ($\delta_{-i,q}$) – the greatest possible amount of production (consumption) of product q in node i.

To operate with the data of one type, we shall introduce a fake site 0 and we shall perform a substitution of variables: $f_{m+_i,q} = x_{-i,q}$, $c_{m+_i,q} = h_{-i,q}$, $d_{m+_i,q} = \delta_{-i,q}$ and $f_{m+_2i,q} = x_{i,q}$, $c_{m+_2i,q} = h_{i,q}$, $d_{m+_2i,q} = \delta_{i,q}$ where arcs $(i,0)$ have numbers $m+i$, and arc $(0,i)$ – number $m+2i$, $i = \overline{1,n}$.

We have received the net $S = \{I, U_0\}$ with a new set of arcs $U_0 = U \cup \{(i,0),(0,i)\}$, that should consist only of those arcs, for which $\min\Big(d_k, \sum_{q=1}^{p} d_{k,p}\Big) > 0$, and $m_0 = |U_0|$. For the new net we shall supplement lists of initial and final clusters O and T.

So, we have received a following mathematical model:

$$H(f) = \sum_{q=1}^{p}\sum_{k=1}^{m_0} c_{k,q} f_{k,q} \to \min$$

$$\sum_{i=O(k)} f_{k,q} - \sum_{i=T(k)} f_{k,q} = a_{i,q}, \quad i = \overline{1,n},\ q = \overline{1,p},$$

$$\sum_{q=1}^{p} f_{k,q} \le d_k, \quad 0 \le f_{k,q} \le d_{k,q},\ k = \overline{1,m_0}.$$

It was proved in papers [1,2] that these two classes of problems: minimax problem of distribution program and inhomogeneous dynamic network flow problem can be reduced to a network optimization problem of a special structure.

1. Problem definition

We consider a finite directed connected network $S = \{I, U\}$ with node set I and arc set U, defined at $I \times I$, $(|I| < \infty, |U| < \infty)$. Let I consist of nodes $i \in I^c$ with constant intensity a_i and nodes $i \in I^d = I \setminus I^c$ with variable intensity. Each arc $(i,j) \in U$ has been assigned three numbers $c_{ij} \in R$, $\mu_{ij} \in R \setminus \{0\}$, $d_{ij} \in R \setminus \{0\}$, that we call traffic rate along the arc (i,j), loss coefficient of the traffic along arc (i,j) and capacity of arc (i,j) respectively. In the same way each node $i \in I^d$ is assigned numbers $c_i \in R$, sign$[i] \in \{-1, 1\}$, $a_{*i} \in R$, $a_i^* \in R$, $a_{*i} < a_i^*$, $a_{*i} \geq 0$, that we call production costs at node i, sign of node i, minimum allowable intensity and maximum allowable intensity of node i respectively.

We refer the number equal to product of loss coefficients of forward arcs divided by product of loss coefficients of backward arcs as cycle determinant:

$$D = \prod_{(i,j)\in U_c^+} \mu_{ij} \Big/ \prod_{(i,j)\in U_c^-} \mu_{ij}$$

where U_c^+ stands for forward net arcs set, U_c^- stands for backward net arcs set.

If the cycle determinant is different from one, the cycle can be referred to as non-degenerate, and otherwise as degenerate. Let $D(i_1, i_m), D_{i_1,i_m}$ or $D(i_1, i_2, \cdots, i_m)$ stand for the determinant of circuit $L(i_1, i_m)$ (according to the context).

Let us refer $i \in I^d$ as dynamic nodes and $i \in I^c$ as static nodes. Consider the following problem:

$$\sum_{(ij)\in U} c_{ij} \cdot x_{ij} + \sum_{i\in I^d} c_i \cdot x_i \to \min, \tag{1}$$

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} \mu_{ji} \cdot x_{ji} = \begin{cases} a_i, & i \in I^c, \\ x_i \cdot \text{sign}[i], & i \in I^d; \end{cases} \tag{2}$$

$$0 \leq x_{ij} \leq d_{ij},\ d_{ij} > 0,\ \mu_{ij} \neq 0,\ (i,j) \in U,$$

$$a_{*i} \leq x_i \leq a_i^*,\ a_{*i} \geq 0, a_i^* > a_{*i},\ i \in I^d, \tag{3}$$

$$I_i^+(U) = \{j : (i,j) \in U\},\ I_i^-(U) = \{j : (j,i) \in U\}.$$

We call problem (1)-(3) a productive-transport problem at a generalized network. Let us assume that the net S is connected and $I^d \neq \oslash$.

The aggregate of numbers $x = \{x_i, x_{ij} | (i \in I^d) \text{ and } ((i,j) \in U)\}$ formed from the production plan $\{x_i | (i \in I^d)\}$ and flow $\{x_{ij} | ((i,j) \in U)\}$ we call a feasible solution in a generalized net, if it satisfies the system (2)-(3). Let us refer to a generalized net as a net.

We call equations (2) the general constraints of the problem (1)- (3) and we call equations (3) the simple constraints of the problem (1)-(3).

2. Net support

We refer to the set $S_{on} = \{I_{on}, U_{on}\}, I_{on} \subseteq I^d, U_{on} \subseteq U$, as the support of net of problem (2)-(3) if system

$$\sum_{j \in I_i^+(U_{on})} x_{ij} - \sum_{j \in I_i^-(U_{on})} \mu_{ji} \cdot x_{ji} = \begin{cases} 0, & i \in I \setminus I_{on}, \\ x_i \cdot \text{sign}[i], & i \in I_{on}, \end{cases} \qquad (4)$$

has the only solution $x \equiv 0$ for this set, but accepts the nonzero solution at the following sets: $\{I_{on}, U_{on} \cup (i_0, j_0)\}$ where $(i_0, j_0) \in U \setminus U_{on}$, $\{I_{on} \cup i_0, U_{on}\}$ where $i_0 \in I^d \setminus I_{on}$.

Matrix of the system (4) is obviously the sub-matrix of system (2), and I_{on} and U_{on} are sets corresponding to certain columns of the matrix of system (2). Let us denote the matrix of system (4) as $A_{on} = \{a_{ij}\}$, $i = \overline{1, |I|}$, $j = \overline{1, |I_{on}| + |U_{on}|}$.

Theorem (Support criterion). *The set $S_{on} = \{I_{on}, U_{on}\}, I_{on} \subseteq I^d, U_{on} \subseteq U$, is the net $S = \{I, U\}$ support if and only if*

1) $|I_{on}| + |U_{on}| = |I|$;

2) *each connected component $S_{on}^k = \{I(U_{on}^k), U_{on}^k\}, k = \overline{1, q}$ of the net $\overline{S_{on}} = \{I(U_{on}), U_{on}\}$ is the net of one of the following types:*

a) does not contain a cycle and there is only one dynamic node;

b) does not contain dynamic nodes and there is only one non-degenerate cycle.

3. Optimality criterion

Let $\{x, S_{on}\}$ be the support feasible solution. Let assign nodes $i \in I$ numbers u_i, so that the following equations hold:

$$\begin{cases} -u_i \cdot \text{sign}[i] = c_i, & i \in I_{on}, \\ u_i - \mu_{ij} \cdot u_j = c_{ij}, & (i, j) \in U_{on}. \end{cases} \qquad (5)$$

It follows from the support properties, that the matrix of system (5) is transposed A_{on}. We call $u_i, i \in I$ the net nodes potentials. Get the estimate $\Delta_{ij} = u_i - \mu_{iij} \cdot u_j - c_{ij}$ for each arc $(i, j) \in U \setminus U_{on}$ and the estimate $\Delta_i = -u_i \cdot \text{sign}[i] - c_i$ for each node $u_i, i \in I^d \setminus I_{on}$. Let $\overline{x}$ be the other feasible solution at the net under consideration and $\Delta x = \overline{x} - x$ is the flow x increment.

It follows from (2) that

$$\sum_{j \in I_i^+(U)} \Delta x_{ij} - \sum_{j \in I_i^-(U)} \mu_{ji} \cdot \Delta x_{ji} = \begin{cases} 0, & i \in I^c, \\ \Delta x_i \cdot \text{sign}[i], & i \in I^d. \end{cases} \qquad (6)$$

Let us get $\Delta f = f(\overline{x}) - f(x)$:

$$\Delta f_{(\Delta x)} = \sum_{(i,j)\in(U)} c_{ij} \cdot \Delta x_{ij} + \sum_{i\in I^d} c_i \cdot \Delta x_i$$
$$= -\sum_{(i,j)\in U\setminus U_{on}} \Delta_{ij} \cdot \Delta x_{ij} - \sum_{i\in I^d\setminus I_{on}} \Delta_i \cdot \Delta x_i. \quad (7)$$

Let U_n stand for the set $U \setminus U_{on}$, and I_n for the set $I^d \setminus I_{on}$.

Theorem (Optimality criterion). *The conditions*

$$\begin{array}{ll} \Delta_{ij} \leq 0 & \text{if } x_{ij} = 0,\ \Delta_{ij} \geq 0, \text{ if } x_{ij} = d_{ij}, \\ \Delta_{ij} = 0 & \text{if } 0 \leq x_{ij} \leq d_{ij}, \text{ if } (i,j) \in U_n \text{ for the arcs,} \\ \Delta_i \leq 0 & \text{if } x_i = a_{*i},\ \Delta_i \geq 0 \text{ if } x_i = a_i^*, \\ \Delta_i = 0 & \text{if } a_{*i} \leq x_i \leq a_i^*, \text{ if } i \in I_n \text{ for the nodes} \end{array} \quad (8)$$

are sufficient and, in the case when the feasible support solution is non-degenerate, necessary for the support feasible solution $\{x, S_{on}\}$ to be optimal.

4. Method iteration

Let $\{x, S_{on}\}$ be the support non-degenerate feasible solution, that does not satisfy optimality criterion conditions (8). As it follows from formula (7), feasible solution x can be improved. Let Δ_{st} be the dominant estimate for the non-support arcs, that does not satisfy conditions (8), and let Δ_r be the dominant estimate for those non-support nodes, where the conditions (8) do not hold true. We call the aggregate of numbers $l = (l_i, i \in I^d, l_{ij}, (i,j) \in U)$ a direction, if it satisfies the system of linear equations (6). It is obvious that if x is a feasible solution, then $x + l = (x_i + l_i, x_{ij} + l_{ij} | i \in I^d, l_{ij}, (i,j) \in U)$ satisfies the general constraints of the problem (1) - (3). Consider algorithm for the determination of the certain feasible direction l and the maximum possible $\ominus^0 \geq 0$ satisfying the following: $x + \Theta^0 \cdot l = (x_i + \Theta^0 \cdot l_i, x_{ij} + \Theta^0 \cdot l_{ij} | i \in I^d, l_{ij}, (i,j) \in U)$ is the feasible solution.

In this case, as it follows from (7), the objective function fluctuation will be maximal for the given l, if $\overline{x} = x + \Theta^0 \cdot l$.

Let

$$\Delta_0 = \begin{cases} \Delta_{st}, & \text{if } |\Delta_{st}| \geq \Delta_r, \\ \Delta_r, & \text{if } |\Delta_{st}| < \Delta_r \end{cases}$$

and

$$S_{vv} = \begin{cases} \{(s,t)\}, & \text{if } |\Delta_0| = \Delta_{st}, \\ \{r\}, & \text{if } |\Delta_0| = \Delta_r. \end{cases}$$

The iteration method consists of the following steps:

Step 1. Define potentials and determine estimates for the arcs $(i,j) \in U_n$ and nodes $i \in I_n$.

Step 2. Define Δ_0.

Step 3. Define the feasible direction l based on the conditions:

a) $l_{ij} = 0, (i,j) \in U_n \setminus S_{vv}$ and $l_i = 0, i \in I_n \setminus S_{vv}$

b) if $\Delta_0 = \Delta_{st}$, then $l_{st} = \operatorname{sign}(\Delta_0)$ or, if $\Delta_0 = \Delta_r$, then $l_r = \operatorname{sign}(\Delta_0)$.

It follows from the optimality criterion, that $l \neq 0$ always exists.

Step 4. Define Θ^0 as it follows:

$$\Theta^0 = \min\Big[\min_{((i,j)\in U_{on}\cup S_{vv}),(l_{ij}>0))}\Big(\frac{d_{ij}-x_{ij}}{l_{ij}}\Big), \min_{((i,j)\in U_{on}\cup S_{vv}),(l_{ij}<0))}\Big(\frac{-x_{ij}}{l_{ij}}\Big),$$
$$\min_{((i\in I_{on}\cup S_{vv}),(l_{ij}>0))}\Big(\frac{a_i^*-x_i}{l_i}\Big), \min_{((i\in I_{on}\cup S_{vv}),(l_{ij}<0))}\Big(\frac{a_{*i}-x_i}{l_i}\Big)\Big].$$

Let S_{viv} stand for the set, which consists only of the arc or the node where Θ^0 is attained.

Step 5. Change feasible solution x as follows:

$$\begin{cases} \overline{x}_{ij} = x_{ij} + l_{ij}\cdot\Theta^0, & (i,j) \in (U_{on} \cup S_{vv}), \\ \overline{x}_i = x_i + l_i \cdot \Theta^0, & i \in (I_{on} \cup S_{vv}). \end{cases}$$

Step 6. Re-construct the support. New support $S_{cor} = (S_{on} \cup S_{vv}) \setminus S_{viv}$.

5. Algorithm for the feasible direction derivation

Let $S_{cor} = \{I_{cor}, U_{cor}\}$. It follows from net $S_{on} = \{I(U_{on}), U_{on}\}$ analysis that the net $S_{cor} = \{I(U_{cor}), U_{cor}\}$ contains the only connected component, and its structure differs from those described in the optimal criterion.

This connected component can belong to one of the following types:

a) contains 2 dynamic nodes and does not contain cycles.

b) contains 1 dynamic node and 1 non-degenerate cycle.

c) contains at least 2 cycles, and one of them is non-degenerate.

It comes to the case a) when we add an arc bridging two connected components, that contain dynamic nodes of the support set, or when we add the dynamic node included into the connected component, which contains a dynamic node of the support set.

It comes to the case b) when we add a dynamic node included into the connected component, which contains a cycle or when we add an arc bridging connected components, one of which contains the only cycle, other contains a dynamic node of the support set or when we add an arc with both nodes belonging to the connected component, which contains a dynamic node to the support.

The case c) appears, when we add an arc bridging connected components with cycles, or an arc with both nodes belonging to the connected component, which contains a cycle of the support set.

It is obvious that vector l components are equal to zero for the corresponding arcs and nodes, that satisfy the conditions a) and b) of the support criterion, from the support connected components. Also, when considering cases a), b) and c), we can cast isolated arcs out, as we can prove that components l, that correspond to these arcs are equal to zero. Let us further consider, that casting-out operation on the isolated arcs has already been carried out.

Consider algorithms of the feasible direction derivation in details.

Case a)

Find circuit $L(i_0, j_0)$ bridging two dynamic nodes i_0 and j_0. The circuit always exists, because these nodes belong to the same connected component. Let $l(L_k)$ stand for the feasible direction of the feasible solution fluctuation at k-th arc of $L(i_0, j_0)$, and $\mu(L_k) = \mu_{ij}$, where i is the initial node of arc $L_k(i_0, j_0)$, j is the final node of the arc $L_k(i_0, j_0)$.

Then feasible directions of the feasible solution fluctuation are determined from the following recursion relations:

$$l(L_1) = 1;\ l(L_{k+1}) = \begin{cases} \mu_k \cdot l(L_k), & \text{if } L_k \text{ and } L_{k+1} \in L^+, \\ -\dfrac{\mu_k \cdot l(L_k)}{\mu_{k+1}}, & \text{if } L_k \in L^+ \text{ and } L_{k+1} \in L^-, \\ -l(L_k), & \text{if } L_k \in L^- \text{ and } L_{k+1} \in L^+, \\ \dfrac{l(L_k)}{\mu_{k+1}}, & \text{if } L_k \text{ and } L_{k+1} \in L^-, \end{cases}$$
$$k = \overline{1, |L(i_0, j_0)| - 1}.$$

Feasible directions for the dynamic nodes i_0 and j_0 are defined as follows:

$$l_{i_0} = \begin{cases} \text{sign}[i_0] \cdot l(L_1), & \text{if } L_1(i_0, j_0) \in L^+, \\ -\text{sign}[i_0] \cdot \mu(L_1) \cdot l(L_1), & \text{if } L_1(i_0, j_0) \in L^-, \end{cases}$$

$$l_{j_0} = \begin{cases} -\text{sign}[j_0] \cdot \mu(L_{|L(i_0,j_0)|}) \cdot l(L_{|L(i_0,j_0)|}), & \text{if } L_{|L(i_0,j_0)|} \in L^+, \\ \text{sign}[j_0] \cdot l(L_{|L(i_0,j_0)|}), & \text{if } L_{|L(i_0,j_0)|} \in L^-, \end{cases}$$

It is easy to see that above defined feasible directions of the feasible solution fluctuation fit general constraints of the problem.

Case b)

Connected component has the structure represented on Figure 1.

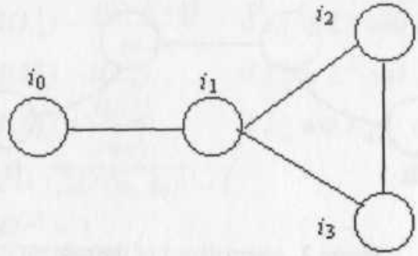

Figure 1. Structure of the connected component of type b).

Consider the node i_0 to be dynamic. Find circuit $L_(i_0, j_0)$ containing the nodes $(i_0, i_1, i_2, i_3, i_1, i_0)$. Feasible directions of the feasible solution's fluctuation are defined with the help of the following recursion relations in two stages:

$$l(L_k) = 0; \quad k = \overline{1, |L(i_0, j_0)|},\ l_{i_0} = 0.$$

$$l(L_1) = 1;\ l(L_{k+1}) = \begin{cases} l(L_k) + \mu_k \cdot l(L_k), & \text{if } L_k \text{ and } L_{k+1} \in L^+, \\ l(L_k) - \dfrac{\mu_k \cdot l(L_k)}{\mu_{k+1}}, & \text{if } L_k \in L^+ \text{ and } L_{k+1} \in L^-, \\ l(L_k) - l(L_k), & \text{if } L_k \in L^- \text{ and } L_{k+1} \in L^+, \\ l(L_k) + \dfrac{l(L_k)}{\mu_{k+1}}, & \text{if } L_k \text{ and } L_{k+1} \in L^-, \end{cases}$$

$$k = \overline{1, |L(i_0, j_0)| - 1}.$$

$$l_{i_0} = \begin{cases} \operatorname{sign}[i_0] \cdot l(L_1), & \text{if } L_1(i_0, j_0) \in L^+, \\ -\operatorname{sign}[i_0] \cdot \mu(L_1) \cdot l(L_1), & \text{if } L_1(i_0, j_0) \in L^-. \end{cases}$$

It is easy to see that the feasible directions defined above fit general constraints of the problem.

Case c)

Case c) has two alternatives: two cycles without common arcs (Figure 2) or two cycles with common arcs (Figure 3).

Consider the 1-st alternative.

Let (i_0, j_0) be the added arc, and $L^1(i_0, j_0)$ is the circuit containing the nodes $(i_0, i_2, i_3, i_4, i_5, i_6, i_7, i_4, i_3, j_0, i_0)$, and denote the determinant of this circuit as $D^{(1)}$, $L^2(i_0, j_0)$ is the circuit containing the nodes $(i_0, i_2, i_3, j_0, i_0)$, denote the determinant of this circuit as $D^{(2)}$.

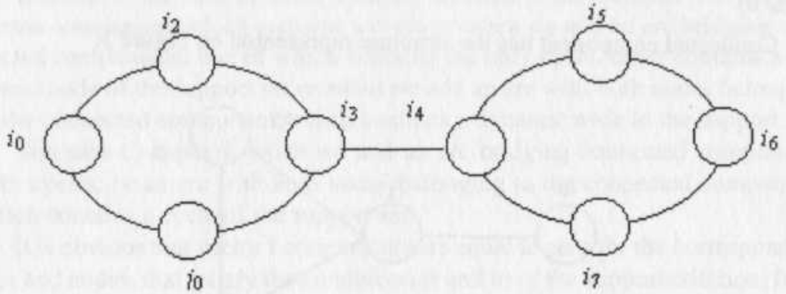

Figure 2. Alternative 1 of the case c).

We can determine feasible directions with the help of the following algorithm. If we accept that $l_{i_0,j_0} = y$, then while computing $l(L_k)$ for the rest of arcs of the circuit $L^1(i_0, j_0)$ in accordance with the formulas of the case b) we get $l_{i_0,j_0} = D^{(1)} \cdot y$. Alike for circuit $L^2(i_0, j_0)$, accepting $l_{i_0,j_0} = z$, we get $l_{i_0,j_0} = D^{(2)} \cdot z$.

It is obvious that if we sum up the feasible fluctuations at the arc we will get again the feasible fluctuation. Thus, to get the feasible fluctuation for the net we should solve the following system:

$$\begin{cases} D^{(1)} \cdot y + D^{(2)} \cdot z = z + y \\ z + y = 1 \end{cases} \leftrightarrow \begin{cases} y = (1 - D^{(2)})/(D^{(1)} - D^{(2)}) \\ z = (D^{(1)} - 1)/(D^{(1)} - D^{(2)}). \end{cases}$$

Feasible directions are defined with the help of the following recursive relations:

$$l(L_k) = 0;\ k = \overline{1, |L^1(i_0, j_0)|}, \quad l(L_k) = 0;\ k = \overline{1, |L^2(i_0, j_0)|}.$$

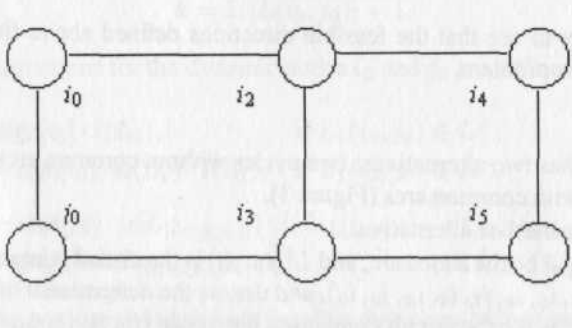

Figure 3. Alternative 2 of the case c).

$$l(L_1^1) = \frac{1 - D^{(2)}}{D^{(1)} - D^{(2)}};$$

$$l(L_{k+1}^1) = \begin{cases} l(L_k^1) + \mu_k \cdot l(L_k^1), & \text{if } L_k^1 \text{ and } L_{k+1}^1 \in L^+, \\ l(L_k^1) - \dfrac{\mu_k \cdot l(L_k^1)}{\mu_{k+1}}, & \text{if } L_k^1 \in L^+ \text{ and } L_{k+1}^1 \in L^-, \\ l(L_k^1) - l(L_k^1), & \text{if } L_k^1 \in L^- \text{ and } L_{k+1}^1 \in L^+, \\ l(L_k^1) + \dfrac{l(L_k^1)}{\mu_{k+1}}, & \text{if } L_k^1 \text{ and } L_{k+1}^1 \in L^-, \end{cases}$$

$$k = \overline{1, |L^1(i_0, j_0)| - 1}.$$

$$l(L_1^2) = \frac{D^{(1)} - 1}{D^{(1)} - D^{(2)}};$$

$$l(L_{k+1}^2) = \begin{cases} l(L_k^2) + \mu_k \cdot l(L_k^2), & \text{if } L_k^2 \text{ and } L_{k+1}^2 \in L^+, \\ l(L_k^2) - \dfrac{\mu_k \cdot l(L_k^2)}{\mu_{k+1}}, & \text{if } L_k^2 \in L^+ \text{ and } L_{k+1}^2 \in L^-, \\ l(L_k^2) - l(L_k^2), & \text{if } L_k^2 \in L^- \text{ and } L_{k+1}^2 \in L^+, \\ l(L_k^2) + \dfrac{l(L_k^2)}{\mu_{k+1}}, & \text{if } L_k^2 \text{ and } L_{k+1}^2 \in L^-, \end{cases}$$

$$k = \overline{1, |L^2(i_0, j_0)| - 1}.$$

$D^{(1)} \neq D^{(2)}$, that can be easily derived from the following:

$$\frac{D^{(1)}}{D^{(2)}} = \frac{D(i_0, i_2, i_3) \cdot D(i_3, i_4) \cdot D(i_4, i_5, i_6, i_7, i_4) \cdot D(i_4, i_3) \cdot D(i_3, j_0, i_0)}{D(i_0, i_2, i_3, j_0, i_0)}$$

$$= \frac{D(i_0, i_2, i_3, j_0, i_0) \cdot D(i_4, i_5, i_6, i_7, i_4)}{D(i_0, i_2, i_3, j_0, i_0)} = D(i_4, i_5, i_6, i_7, i_4) \neq 0$$

as the cycle $(i_4, i_5, i_6, i_7, i_4)$ is non-degenerate.

It is easy to see that the formulas and the relations for Alternative 2 are just the same as the formulas and relations for Alternative 1.

And it is easy to see that above defined feasible directions fit general constraints of the problem.

6. Potentials determination

Let us describe the constructive method for the solution of system (5) solution. In the terms of matrix the system is of the form $u^T \cdot A_{on} = c_{on}^T$. Since $|A_{on}| \neq 0$, the unique solution of this system exists. As it was shown in the proof [3,4] of the support criterion, the support consists of the connected components. Thus system $u^T \cdot A_{on} = c_{on}^T$ falls apart into q independent systems $u^T \cdot A_k = c_k^T$, $k = \overline{1, q}$, (one system for each connected component). As it was shown in the support criterion proof [3,4], the connected components could be either of type a) or b).

a) Current connected component does not contain a cycle and there is the dynamic node i_0. The node i_0 potential is defined as:

$$u_{i_0} = -\text{sign}[i_0] \cdot c_{i_0}.$$

Potentials of the other nodes are defined as follows:

$$\begin{cases} u_i = \mu_{ij} \cdot u_j + c_{ij}, & \text{if we know the potential of the node } j \text{ of arc } (i,j), \\ u_j = \dfrac{u_i - c_{ij}}{\mu_{ij}}, & \text{if we know the potential of the node } i \text{ of arc } (i,j). \end{cases} \qquad (9)$$

It is obvious that within the finite number of steps we can define potentials of all the nodes of the connected component.

b) The connected component contains the only non-degenerate cycle, and does not contain the nodes $i \in I_{on}$. First define the potential of any node of the cycle. Let this node be i_1. The potential of the node is

$$u_{i_1} = \frac{\sum_{p \in I_{cycle} \setminus \{i_1\}} D_{i_1 p} \cdot \overline{c}_{p,p+1}}{1 - D_c}$$

where I_{cycle} is the set of the cycle nodes, D_c is the cycle determinant, $D_{i_1 p}$ is the determinant of the circuit which connects nodes i_1 and p

$$\overline{c}_{p,p+1} = \begin{cases} c_{p,p+1}, & \text{for the forward arcs} \\ -\dfrac{c_{p,p+1}}{\mu_{p,p+1}}, & \text{for the forward arcs.} \end{cases}$$

Potentials of the other nodes are defined according to (9).

The example is considered:

$$\begin{gathered} -z \to \min \\ h_{13} + h_{14} + h_{15} \le 1, \\ h_{23} + h_{24} + h_{25} \le 1, \\ -10 \cdot h_{13} - 4 \cdot h_{23} \le -z, \\ -2 \cdot h_{14} - 5 \cdot h_{24} \le -z, \\ -3 \cdot h_{15} - 6 \cdot h_{25} \le -z, \\ h_{ij} \ge 0 \quad i = \overline{1,2},\ j = \overline{3,5}. \end{gathered}$$

At the generalized network this problem can be represented as follows:

$$
\begin{aligned}
-h_6/3 &\to \min,\\
h_{13}+h_{14}+h_{15} &= h_1,\\
h_{23}+h_{24}+h_{25} &= h_2,\\
h_{36}-10\cdot h_{13}-4\cdot h_{23} &= -h_3,\\
h_{46}-2\cdot h_{14}-5\cdot h_{24} &= -h_4,\\
h_{56}-3\cdot h_{15}-6\cdot h_{25} &= -h_5,\\
-h_{36}-h_{46}-h_{56} &= -h_6,\\
h_{36} &= h_6/3,\\
h_{46} &= h_6/3,\\
h_{56} &= h_6/3,
\end{aligned}
$$

$$h_{ij}\ge 0 \quad (i,j)\in U,\ 0\le h_i\le 1,\ i\in I^1,\ h_i\ge 0,\ i\in I^2,$$

$$I^1=\{1,2\},\ I^2=\{3,4,5,6\},\ i_0=6.$$

This example was solved by the help of algorithm for the solution of a industrial-transport problem with additional constraints in view of network properties of the problem, theoretic-graph specificity of the structure of support.

The value function is equal -3.6, flows on arcs: $h_{13} = 0.36$, $h_{14} = 0.00$, $h_{15} = 0.64$, $h_{23} = 0.00$, $h_{24} = 0.72$, $h_{25} = 0.28$, $h_{36} = 3.6$, $h_{46} = 3.6$, $h_{56} = 3.6$, nodes intensity: $h_1 = 1.00$, $h_2 = 1.00$, $h_3 = 0$, $h_4 = 0$, $h_5 = 0$, $h_6 = 10.8$.

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