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INHOMOGENEOUS NETWORK DISTRIBUTIVE PROBLEM

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On the example of generalized oriented finite network $S=\{I,U\}$, $|I|<\infty$ we consider inhomogeneous (two-product) transport problem with additional constraints

$$\sum_{k\in K}\sum_{(i,j)\in U} c_{ij}^{k} x_{ij}^{k} \longrightarrow \min, \tag{1}$$

$$\sum_{j\in I_i^+(U)} x_{ij}^{k} - \sum_{j\in I_i^-(U)} \mu_{ji}^{k} x_{ji}^{k} = a_i^{k},\ i\in I,\ k\in K=\{1,2\}, \tag{2}$$

$$\sum_{k\in K}\sum_{(i,j)\in U} \lambda_{ij}^{kp} x_{ij}^{k} = \alpha_p,\ p=\overline{1,l}, \tag{3}$$

$$x_{ij}^{k}\geqslant 0,\ k\in K,\ (i,j)\in U,\ x_{ij}^{1}+x_{ij}^{2}\leqslant d_{ij},\ (i,j)\in U, \tag{4}$$

where vector $x_{ij}=(x_{ij}^1, x_{ij}^2)$ defines the value of inhomogeneous stream along the arc $(i,j)\in U$; $a_i=(a_i^1, a_i^2)$ is the vector of node intensity $i\in I$; d_{ij} is the value of the arc throughput $(i,j)\in U$; $c_{ij}=(c_{ij}^1, c_{ij}^2)$ is the cost of a unit arc stream along the arc $(i,j)\in U$; $\mu_{ij}=(\mu_{ij}^1, \mu_{ij}^2)$ is the vector, containing arc factors and reflecting phenomenon of the arc-stream transformation; $I_i^+(U)=\{j:(i,j)\in U\}$, $I_i^-(U)=\{j:(j,i)\in U\}$.

The problem, that is dual to (1) - (4), has the form

$$\sum_{k=1}^{2}\sum_{i\in I} a_i^k u_i^k + \sum_{p=1}^{l} \alpha_p z_p - \sum_{(i,j)\in U} d_{ij} w_{ij} \to \max \tag{5}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{\ell} \lambda_{ij}^{kp} z_p - w_{ij} \le c_{ij}^k, w_{ij} \ge 0, (i,j) \in U, k \in K \quad (6)$$

Let us designate arc set $U_{on} = \{U_{on}^1, U_{on}^2, U^*\}$, $U_{on}^k \subset U^k$, $k=1,2$, $U^* \subset U_{on}^1 \cap U_{on}^2$, as a support of generalized network $S = \{I, U\}$, if the system

$$\sum_{j \in I_i^+(U_{on}^k)} x_{ij}^k - \sum_{j \in I_i^-(U_{on}^k)} \mu_{ji} x_{ji}^k = 0, i \in I, k=1,2,$$

$$\sum_{k=1}^{2} \sum_{(i,j)^k \in U_{on}^k} \lambda_{ij}^{kp} x_{ij}^k = 0, \ p=\overline{1,\ell}, \quad (7)$$

$$x_{ij}^1 + x_{ij}^2 = 0, \ (i,j) \in U^*,$$

has only trivial solution, but has also nontrivial solution for each of the following arc sets:

1) $\{U_{on}^1, U_{on}^2, U^* \setminus (i,j)\}$, where (i,j) is any arc from U^* ;

2) $\{U_{on}^1 \cup (i,j)^1, U_{on}^2, U^*\}$, where $(i,j)^1 \overline{\in} U_{on}^1$;

3) $\{U_{on}^1, U_{on}^2 \cup (i,j)^2, U^*\}$, where $(i,j)^2 \overline{\in} U_{on}^2$.

The pair $\{x, U_{on}\}$ from the stream and support is the support stream. The support stream $\{x, U_{on}\}$ is said to be nondegenerate if the following unequalities are fulfilled $x_{ij}^k > 0$, $(i,j) \in U_{on}^k$, $k=1,2$; $x_{ij}^1 + x_{ij}^2 < d_{ij}$, $(i,j) \in U_{on}^k \setminus U^*$.

By the support U_{on} we calculate node potentials $u_i = (u_i^k, k \in K)$, $i \in I$, potentials $z_p, p=1,\ell$ of additional constraints. With the help of these potentials we'll calculate the estimates of nonsupport arcs $(i,j)^k \in U_H^k$, $U_H^k = U_\ell^k \setminus U_{on}^k$ and arcs $(i,j) \in U^*$:

$$\Delta_{ij}^k = u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{\ell} \lambda_{ij}^{kp} z_p - c_{ij}^k, \ k \in K.$$

Criterium of optimality. The relations

$$\Delta_{ij}^k \le 0 \quad \text{for } x_{ij}^k = 0; \ \Delta_{ij}^k = 0 \quad \text{for } x_{ij}^k > 0$$

on unsaturated arcs $(x_{ij}^1 + x_{ij}^2 < d_{ij})$;

$$\Delta_{ij}^1 = \Delta_{ij}^2 \ge 0 \quad \text{for } x_{ij}^1 > 0, \ x_{ij}^2 > 0; \quad (8)$$

$$\Delta_{ij}^\ell \ge \Delta_{ij}^k, \Delta_{ij}^\ell \ge 0, \ell \ne k, \ell, k \in K \text{ for } x_{ij}^\ell = d_{ij}, \ x_{ij}^k = 0$$

on saturated arcs $(x_{ij}^1 + x_{ij}^2 = d_{ij})$ are sufficient, and in the case of nondegeneracy they are necessary for multistream x optimality.

Sufficient condition of suboptimality. Let for the support stream $\{x, U_{on}\}$ the following unequality be fulfilled

$$-\sum_{\substack{(i,j) \in U, \ k \in K \\ w_{ij}=0}} \Delta_{ij}^k x_{ij}^k - \sum_{\substack{(i,j) \in U \\ w_{ij} = \Delta_{ij}^k > 0, \ \ell, k \in K, \ \ell \ne k}} (\Delta_{ij}^\ell x_{ij}^\ell + \Delta_{ij}^k (x_{ij}^k - d_{ij}) \le \delta, \quad (9)$$

where $w_{ij} = max\{\Delta_{ij}^k, k \in K; 0\}$.

Then x is δ - optimal stream. For every δ -optimal stream x^δ there exists such a support U_{on}, that suboptimality estimate β of the support stream $\{x^\delta, U_{on}\}$ satisfies the unequality $\beta \le \delta$

The suboptimality (9) suggested allows us to end the solution process after the required accuracy of approximation to optimal stream, taking into account the cost function, is reached.

Let us assume, that initial stream x does not satisfy the optimality criterium (8), and that for a set δ the unequality (9) is not fulfilled. Iteration for the stream x improvement is the following: among the arcs for which relations (8) are not realized, let us define an arc with maximal violation of optimality criterium. The following cases are possible:

1) maximal violation of optimality criterium equals to difference of the estimates $\Delta_{ij}^k - \Delta_{ij}^\ell$, $\ell, k = 1,2$, $\ell \ne k$;

2) maximal violation of optimality criterium has the form Δ_{ij}^k and is reached on the arc (i_0, j_0)

a) $(i_0, j_0) \in U^*$;

b) $(i_0, j_0) \overline{\in} U^*$.

Considering each case, it is easy to build an appropriate direction $y = \{y_{ij}^k, (i,j)^k \in U^k, k \in K\}$ of inhomogeneous stream x improvement. Maximally admissible step θ in chosen direction is defined from the condition of direct-constraints (4) fulfilment. Iteration ends in construction of a new stream $\bar{x} = x + \theta y$. Direct method suggested is finite, if support stream $\{x, U_{on}\}$ is nondegenerate.

Dual method is intended for receiving optimal or δ -optimal streams of the problem (1) - (4) with transformation of dual plans.

Vector $\eta = (u^k, k \in K; z, w)$, on which the constraints of the problem (5) - (6) are fulfilled is said to be dual plan.

Dual plan η we designate as coordinated with the costream $\delta = (\delta_{ij}^1, \delta_{ij}^2)$, $\delta_{ij}^k = u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{\ell} \lambda_{ij}^{kp} z_p - c_{ij}^k$, $k=1,2$, if the relation $w_{ij} = max\{0, \delta_{ij}^1, \delta_{ij}^2\}$ is fulfilled. The pair $\{x, U_{on}\}$ from the costream and network support S is said to be support costream. Let us assume, that initial support costream $\{\delta, U_{on}\}$ is constructed according to the coordinated with it dual plan η .

We'll construct corresponding pseudostream in the following way:

$$x_{ij}^{k}=0,\ \text{if}\ \delta_{ij}^{k}\leqslant 0\ \text{or}\ \delta_{ij}^{k}\leqslant w_{ij},\ (i,j)\in U_{H}^{k} \tag{10}$$
$$x_{ij}^{k}=d_{ij},\ x_{ij}^{k}=0\ \text{if}\ w_{ij}=\delta_{ij}^{\ell}>0,\ (i,j)\in U_{H}^{k},\ (i,j)\in U_{H}^{\ell},\ \ell\neq k.$$

Support arc pseudostreams are unambiguously defined from conditions (2). Let us calculate an increment of dual cost function

$$\rho=\sum_{k=1}^{2}\sum_{i\in I} a_i^k\,\Delta u_i^k+\sum_{p=1}^{\ell} d_p\,\Delta z_p-\sum_{(i,j)\in U} d_{ij}\,\Delta w_{ij}=$$
$$=\sum_{k=1}^{2}\sum_{i\in I}\Delta u_i^k\Big(\sum_{j\in I_i^+(U)} x_{ij}^k-\sum_{j\in I_i^-(U)}\mu_{ji}^k x_{ji}^k\Big)+$$
$$+\sum_{p=1}^{\ell}\Delta z_p\sum_{k=1}^{2}\sum_{(i,j)\in U}\lambda_{ij}^{kp}x_{ij}^{k}=$$
$$=\sum_{k=1}^{2}\sum_{(i,j)\in U} x_{ij}^k\,\Delta\delta_{ij}^k-\sum_{(i,j)\in U} d_{ij}\,\Delta w_{ij}$$

We designate $\delta_{ij}^{k_0(i,j)}=\max\{\delta_{ij}^{k},\ k\in K(i,j)\}$, $\tilde\delta_{ij}^{k_1(i,j)}=$
$=\max\{\tilde\delta_{ij}^{k},\ k\in K(i,j)\}$, $(i,j)\in U$, $\tilde\delta_{ij}^{k}=\delta_{ij}^{k}+\Delta\delta_{ij}^{k}$;
$U_0^1=\{(i,j)\in U: w_{ij}=0\}$, $U_0^2=\{(i,j)\in U: k_0(i,j)\in K_{on}(i,j)\}$,
$U_0^3=\{(i,j)\in U: k_0(i,j)\in K_H(i,j)\}$.

With the allowance for the values of nonsupport arc pseudostreams (10) and coordination conditions, the formula of incrementation will take the form

$$\rho=\sum_{k=1}^{2}\sum_{(i,j)\in U_{on}} x_{ij}^k\,\Delta\delta_{ij}^k+\sum_{\substack{(i,j)\in U_0^2,\\ \tilde\delta_{ij}^{k_1(i,j)}\geqslant 0}} d_{ij}\big(\delta_{ij}^{k_0(i,j)}-\delta_{ij}^{k_1(i,j)}\big)+$$
$$+\sum_{\substack{(i,j)\in U_0^3,\\ \tilde\delta_{ij}^{k_1(i,j)}\geqslant 0}} d_{ij}\big(\tilde\delta_{ij}^{k_0(i,j)}-\tilde\delta_{ij}^{k_1(i,j)}\big)-\sum_{\substack{(i,j)\in U_0^1,\\ \tilde\delta_{ij}^{k_1(i,j)}>0}} d_{ij}\,\tilde\delta_{ij}^{k_1(i,j)}+$$
$$+\sum_{\substack{(i,j)\in U_0^2,\\ \tilde\delta_{ij}^{k_1(i,j)}<0}} d_{ij}\,\delta_{ij}^{k_0(i,j)}+\sum_{\substack{(i,j)\in U_0^3,\\ \tilde\delta_{ij}^{k_1(i,j)}<0}} d_{ij}\,\tilde\delta_{ij}^{k_0(i,j)}$$

Sufficient condition of suboptimality. If $\beta\leqslant\varepsilon$, where

$$\beta=-\sum_{(i,j)\in U_0^1}\sum_{k\in K_{on}(i,j)} x_{ij}^k\delta_{ij}^k-\sum_{(i,j)\in U_0^2}\Big(x_{ij}^k\delta_{ij}^k+$$
$$+\delta_{ij}^{k_0(i,j)}\big(x_{ij}^{k_0(i,j)}-d_{ij}\big)\Big),\ k\neq k_0(i,j),\ k\in K,\ k_0(i,j)\in K,$$

and support arc pseudostreams $x=\{x_{ij}^k,\ (i,j)\in U,\ k\in K\}$ satisfy direct constraints of the problem (1) - (4)

$$x_{ij}^k\geqslant 0,\ (i,j)\in U,\ k\in K,\ x_{ij}^1+x_{ij}^2\leqslant d_{ij},\ (i,j)\in U$$

then x is ε-optimal stream.

Let $\{\delta,\ U_{on}\}$ be a support costream for which the sufficient condition of suboptimality is not fulfilled. We'll construct a new costream $\tilde\delta$ in the form $\tilde\delta=\delta+\sigma t$, where $t=\{t_{ij}^k,\ (i,j)\in U,\ k\in K\}$ is corresponding direction of the changes in costream δ, σ is a maximally admissible step. The principles for construction of suitable direction t are analogical to the principle for construction suitable direction $y=\{y_{ij}^k,\ (i,j)\in U,\ k\in K\}$. Maximally admissible step σ is calculated by the formula

$$\sigma=\min\{\sigma_{i_0j_0}^{k_0},\ \sigma_{i_1j_1}\},\ \sigma_{i_0j_0}^{k_0}=\min\{\sigma_{ij}^{k},$$
$$k\in K(i,j),\ (i,j)\in U\},\ \sigma_{i_1j_1}=\min\{\sigma_{ij},\ (i,j)\in U\},$$

$$\sigma_{ij}^k=\infty,\ \text{if}\ \delta_{ij}^k t_{ij}^k>0$$
$$\sigma_{ij}^k=-\delta_{ij}^k/t_{ij}^k \quad \text{- otherwise}$$
$$\sigma_{ij_k}=\big(\delta_{ij}^{k_0(i,j)}-\delta_{ij}^{k}\big)/\big(t_{ij}^{k}-t_{ij}^{k_0(i,j)}\big),\ \text{if}$$
$$t_{ij}^{k}-t_{ij}^{k_0(i,j)}>0,\ \delta_{ij}^{k_0(i,j)}>0;$$

$\delta_{ij} = \infty$ - otherwise

Iteration ends with the construction of a new support costream $\{\tilde{\delta}, \tilde{U}_{оп}\}$

The support costream $\{\delta, U_{оп}\}$ is said to be degenerate, if one of the following properties is fulfilled:

1) there exists such an arc $(i,j) \in U_{H}^{K}$ on which $w_{ij} = \delta_{ij}^{K} = 0$;

2) there exists the arc $(i,j) \in U_{H}^{K_1} \cap U_{H}^{K_2}$ or $(i,j) \in U_{оп}^{K_1} \cap U_{H}^{K_2}$, $w_{ij} = \delta_{ij}^{K_1} = \delta_{ij}^{K_2}$, $w_{ij} > 0$;

3) such an arc $(i,j) \in U^*$ will be found, that $w_{ij} = \delta_{ij}^{K} = 0$, $K \in K_{оп}(i,j)$.

In the case of nondegeneracy the algorithm described is finite. The finitness is caused by the finitness of dual support method [1].

References

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