ГАРАНТИРОВАННАЯ ПОЛНАЯ ДОМИНАНТНОСТЬ В ГРАФАХ: СВОЙСТВА И СЛОЖНОСТЬ

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Подмножество вершин S неориентированного графа $G = (V, E)$ представляет собой полностью доминантное множество для G, если каждая вершина G смежна по крайней мере с одной вершиной из S. В этой статье мы рассмотрим гарантированно полностью доминантное множество.

Ключевые слова: гарантированно полностью доминантное множество; алгоритмы аппроксимации; *NP*-полнота.

SECURE TOTAL DOMINATION IN GRAPHS: PROPERTIES AND COMPLEXITY

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A vertex subset S of an undirected graph $G = (V, E)$ is a total dominating set of G if each vertex of G is adjacent to at least one vertex of S. In this paper, we consider secure total dominating sets, i.e., total dominating sets D of G satisfying the following condition: each vertex $v \in V - D$ is adjacent to at least one vertex $u \in D$ with the property that the set $(D - \{u\}) \cup \{v\}$ is total dominating in G. The minimum size of a secure total dominating set of G is the secure total domination number of G.

We present a characterization of secure total dominating sets in (P_5, *bull*)-free graphs and new bounds on the secure total domination number. Besides, we consider a problem of finding this number and provide results on the complexity of this problem in special graph classes.

Keywords: secure total dominating sets; approximation algorithms; *NP*-completeness.

INTRODUCTION

Graph invariants are ones of the central objects of researches in the graph theory. A graph invariant is a number associated with a graph, which is the same for any two isomorphic graphs. Graph invariants arise naturally in mathematical formulations of practical problems and they are considered not only in the field of the graph theory, but also from the algorithmic point of view (exact and approximation algorithms) and the mathematical programming perspective. Ones of the most known and studied graph invariants are the domination and total domination numbers of graphs [1–4]. Recall, a vertex subset S of an undirected graph $G = (V, E)$ is a *dominating* (*total dominating*) *set* of G if each vertex v of the set $V-S$ (respectively, V) is adjacent in G to at least one vertex of S. The minimum size of a dominating (total dominating) set of G is called the domination number (respectively, the total domination number) of G and it is denoted by $\gamma(G)$ (respectively, $\gamma(G)$). Fairly recently, secure versions of (total) dominating sets of a graph have been introduced as graph protection models [5, 6].

Let S be a dominating set of the graph $G = (V, E)$. Consider two vertices $v \in V-S$, $u \in S$. We say that the vertex u *defends* (*totally defends*) the vertex v if the vertices u and v are adjacent and the set $\left(S-\{u\}\right)\cup\{v\}$ is a dominating (respectively, total dominating) set of G. A vertex subset D of G is a *secure dominating* (*secure total dominating*) *set* of G if D is a dominating (respectively, total dominating) set of G and each vertex $v \in V-D$ is defended (respectively, totally defended) by at least one vertex of D. The minimum size of a secure dominating (secure total dominating) set of G is called the *secure domination* (respectively, *secure total domination*) *number* of G and it is denoted by $\gamma_s(G)$ (respectively, $\gamma_{st}(G)$). We note that for every graph there is at least one secure dominating set. Indeed, the set all vertices of a graph is one. At the same time, there are graphs with no secure total dominating sets. It is not hard to see that a graph G has at least one secure total dominating set if and only if G has no isolated vertices. Thus, the secure total domination number is defined solely for graphs with no isolated vertices.

In this paper, we consider the secure total domination number and the problem of finding this number that is formulated as a decision problem as follows:

SECURE TOTAL DOMINATING SET

Instance: A graph $G = (V, E)$ without isolated vertices and an integer k;

Question: Is there a secure total dominating set D of G such that $|D| \le k$? In other words, is $\gamma_{st}(G) \le k$?

The optimization version of the SECURE TOTAL DOMINATING SET problem asks for a minimum cardinality secure total dominating set of a given graph without isolated vertices.

The secure total domination number was introduced by Cockayne et. al. in [5]. This graph parameter is studied in papers [5, 7, 8, 9]. Klostermeyer et. al. in [9] compare the secure (total) domination numbers with other graph parameters and characterize graphs with equal total and secure total domination numbers. Go et. al. in [7, 8] study secure (total) domination numbers in graphs that are obtained by means of the following graph operations: the join, the corona and the composition of graphs. To the best of our knowledge, the computational complexity of the SECURE TOTAL DOMINATING SET problem has not been studied.

BOUNDS ON THE SECURE TOTAL DOMINATION NUMBER

In this section, we give new lower and upper bounds on the secure total domination number.

As usual, $N_G(v)$ denotes the set of all vertices of a graph G adjacent to a vertex v of G and $N_G[v] = N_G(v)\cup\{v\}$. A vertex subset D of G is a *double dominating set* of G if $\left|N_G[v]\cap D\right| \ge 2$ for each vertex v of G. The minimum cardinality of a double dominating set of G is the *double domination number* $\gamma_{\times 2}(G)$ of G.

Lemma 1. $\gamma_{\times 2}(G) \leq \gamma_{st}(G)$ for every graph G without isolated vertices.

We note that if G is a 5-vertex path P_5 or the graph *bull*, then $4 = \gamma_{\times 2}(G) \leq \gamma_{st}(G) = 5$, where the graph *bull* is the graph with vertex set $\{a, b, c, d, e\}$ and the edge set $\{\{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{b, e\}\}$. It is natural and reasonable to figure out graphs G satisfying $\gamma_{\times 2}(G) = \gamma_{st}(G)$.

Lemma 2. Let $G = (V, E)$ be a (P_5, *bull*)-free graph (i. e., a graph without induced subgraphs isomorphic to P_5 or *bull*) without isolated vertices. A vertex subset D of G is a secure total dominating set of G if and only if the set D is a double dominating set of G.

Corollary 3. $\gamma_{st}(G) = \gamma_{\times 2}(G)$ for every (P_5, *bull*)-free graph G without isolated vertices.

We obtain an answer on the following open question posed by Klostermeyer et. al. [9]: is there a graph G such that $\gamma_{st}(G) = 3 \cdot \alpha(G)$, where $\alpha(G) \geq 2$ is the independence number of G? We provide a negative answer to this question by confirming the suspicions of authors of the paper [9] that $\gamma_{st}(G) \leq 2 \cdot \alpha(G)$.

Theorem 4. $\gamma_{st}(G) \leq 2 \cdot \alpha(G)$ for every graph G without isolated vertices.

Now, we consider secure total dominating sets in the composition $G[K_n]$, where K_n is the complete graph on n vertices. Recall that the composition (also known as the lexicographic product or substitution) of two vertex-disjoint graphs G and H is the graph $G[H]$ with the vertex set $VG \times VH$ and $(u_G, u_H) \square (v_G, v_H)$ if and only if either (a) $\{u_G, v_G\} \in EG$ or (b) $u_G = v_G$ and $\{u_H, v_H\} \in EH$. Equivalently, the composition $G[H]$ can be defined as the graph obtained from G and H by taking $|VG|$ copies of H, say H_v, $v \in VG$ and joining by an edge all vertices of H_v with all vertices of H_u if and only if $\{u, v\} \in EG$ [10, 185]. Castillano et. al. in [7] prove that $\gamma_{st}(G[K_n]) \leq \min\left(2 \cdot \gamma(G), \gamma_{st}(G)\right)$ for every connected graph G, any $n > 1$ and conjecture that this upper bound on $\gamma_{st}(G[K_n])$ is actually the exact value of the parameter. We disprove their conjecture by providing a counterexample. We claim that for the graph G depicted in figure the following hold: $2 \cdot \gamma(G) \geq 6$, $\gamma_{st}(G) \geq 6$ and $\gamma_{st}(G[K_n]) \leq 5$ for each $n > 1$.

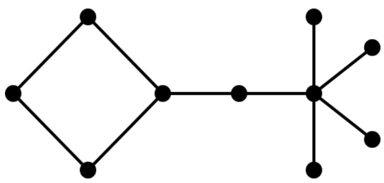

Figure. The graph G that is a counterexample to a conjecture

Thus, for this graph G we obtain that $\gamma_{st}(G[K_n]) < \min\left(2 \cdot \gamma(G), \gamma_{st}(G)\right)$.

COMPLEXITY RESULTS

In this section, we provide some results on the complexity of the SECURE TOTAL DOMINATING SET problem for well-known graph classes. For definitions of graph classes, we refer to [11, 12].

Theorem 5. The SECURE TOTAL DOMINATING SET problem is NP-complete in the following graph classes: split graphs, graphs of separability at most 2, chordal bipartite graphs.

Theorem 6. For any fixed integer $p \geq 4$, the SECURE TOTAL DOMINATING SET problem is NP-complete for planar bipartite graphs with girth at least p and maximum degree at most 3.

Further, we consider the complexity of approximation for the optimization version of the SECURE TOTAL DOMINATING SET problem.

An algorithm is an $\alpha(n)$-approximation algorithm for a minimization problem Π if for each instance x of size n, it computes a solution y of value $m(x, y)$ such that $m(x, y) \leq \alpha(n) \cdot opt(x)$, where $opt(x)$ is the value of an optimum solution for the instance x. If a minimization problem Π admits a polynomial-time $\alpha(n)$-approximation algorithm, we say that the problem Π can be approximated in polynomial time within a factor of $\alpha(n)$; otherwise, we say that it cannot be approximated in polynomial time within a factor of $\alpha(n)$.

Theorem 7. Assuming that $P \neq NP$, SECURE TOTAL DOMINATING SET cannot be approximated in polynomial time within a factor of $c \ln n$, where n is the number of vertices in the input graph and $c < 1$ is a constant. This remains valid even for the problem restricted to chordal graphs.

Theorem 8. SECURE TOTAL DOMINATING SET can be approximated in polynomial time within a factor of $c \ln n$, where n is the number of vertices in the input graph and $c > 1$ is a constant.

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